COMPACTIFYING MODULI OF HYPERELLIPTIC CURVES

MARCO PACINI

ABSTRACT. We construct a new compactification of the moduli space H_g of smooth hyperelliptic curves of genus g. We compare our compactification with other well-known remarkable compactifications of H_g.

1. INTRODUCTION

Let H_g be the moduli space of smooth hyperelliptic curves of genus $g \geq 3$. Several compactifications of H_g have been constructed. For example, there exists a moduli space $\overline{B_m}$ of GIT-semistable binary forms of degree m, where a binary form of degree m is a homogeneous polynomial of degree m in two variables over $\mathbb{C}$, up to non-trivial constants. In particular, $\overline{B_{2g+2}}$ contains H_g as a dense open subset. Recall that a *m-pointed stable curve of genus zero* $(Y, p_1, \dots, p_m)$ is a curve Y of genus 0 with an ordered set of distinct smooth points $p_i \in Y$ such that $|Y_i \cap \overline{Y - Y_i}| + |Y_i \cap \{p_1, \dots, p_m\}| \geq 3$ for every irreducible component Y_i of Y. A *m-marked stable curve of genus zero* is a m-pointed stable curve of genus zero $(Y, p_1, \dots, p_m)$ up to the action of the symmetric group S_m on $p_1, \dots, p_m$. A natural compactification of H_g is given by its closure within the moduli space of Deligne-Mumford stable curves. This compactification is isomorphic to the moduli space $\overline{N_{0,2g+2}}$ of $2g+2$-marked stable curves of genus zero. There exists also a fine moduli space $\overline{M_{0,m}}$ of m-pointed stable curves of genus zero and $\overline{N_{0,m}} = \overline{M_{0,m}}/S_m$.

As pointed out in [AL], $\overline{N_{0,2g+2}}$ and $\overline{B_{2g+2}}$ are different schemes. We construct a compactification $\overline{J_g}$ of H_g, given in term of configurations of plane lines and we compare it to $\overline{N_{0,2g+2}}$ and $\overline{B_{2g+2}}$. Indeed, consider $(C, p_1, \dots, p_{2g+2})$, C a smooth plane conic and $p_i \in C$ distinct points. Pick the $h_g = \binom{2g+2}{2}$ lines spanned by p_i, p_j for $1 \leq i < j \leq 2g+2$. By taking the closure within $Sym^{h_g}(\mathbb{P}^2)^\vee$, we obtain configurations associated to $(C, p_1, \dots, p_{2g+2})$, C singular or $p_i = p_j$. The variety $\overline{J_g}$ is the GIT-quotient of the set of GIT-semistable configurations of lines, with respect to the action of $SL(3)$. A boundary point of $\overline{J_g}$ is a configuration containing at least a non-reduced line. For example, if C is smooth and $p_1 = p_2 \neq p_j$ for $j \geq 3$, the associated configuration contains $\text{span}(p_1, p_j)_{j \geq 3}$ as double lines. The boundary points of $\overline{J_g}$ have the following geometric meaning. If C is smooth and $p_i \in C$ are distinct, consider the double cover $\varphi : X \to C$ branched at the $2g+2$ points p_i. From [ACGH, pag. 288] and [M, Proposition 6.1], we have that $\mathcal{O}_X(\varphi^*(p_i + p_j))$ is a $(g-1)$-th root of ω_X. If $g = 3$, they are the 28 odd theta characteristics of the hyperelliptic curve X. Thus $\overline{J_g}$ is a compactification of H_g given in terms of limits of configurations of higher spin curves of order $g-1$, in the sense of [CCC].

In [H] and [AL], the authors construct a geometrical meaningful morphism $F_g : \overline{N_{0,2g+2}} \to \overline{B_{2g+2}}$. In Theorem 3.3, Theorem 4.5 and Theorem 4.7, we construct rational maps $\overline{N_{0,2g+2}} \overset{\beta_g}{\dashrightarrow} \overline{J_g} \overset{\alpha_g}{\dashrightarrow} \overline{B_{2g+2}}$, giving a factorization of F_g. The construction of α_g follows from Lemma 3.2, proving that it is possible to recover $(C, p_1, \dots, p_{2g+2})$, C smooth conic and $p_i \in C$, from its configuration of lines. In particular, Lemma 3.2 extends the results of [CS1] and [L], stating that a smooth plane quartic can be recovered from its bitangents, to double conics. In fact, the stable reduction of a general one-parameter deformation of a double conic C is an hyperelliptic curve, the double cover of C branched at 8 points. The limits of the bitangents give rise to the configuration of lines associated to C and the 8 points. We point out that a different generalization of bitangents for any genus is theta hyperplanes, used in [CS2] and [GS].

In short, in Section 2 we show properties of twisters of curves. In Section 3, we construct $\overline{J_g}$ and the map α_g. In Section 4, we construct the map β_g, showing that α_g and β_g provide a factorization of F_g.

1.1. **Notation.** We work over $\mathbb{C}$. A *family of curves* is a proper and flat morphism $f\colon \mathcal{C} \to B$ whose fibers are curves. If 0 is a point of a scheme B, set $B^* := B - 0$. A *smoothing* of a curve C is a family $f\colon \mathcal{C} \to B$, where B is a smooth, connected, affine curve of finite type, with a distinguished point $0 \in B$, such that $f^{-1}(0)$ is isomorphic to C and $f^{-1}(b)$ is smooth for $b \in B^*$. A *general smoothing* is a smoothing with smooth total space. Let Y be a scheme and $X \to Y$ be a Y-scheme. Denote by $Sym^m_Y X := X^m_Y / S_m$ the quotient of $X^m_Y = X \times_Y X \times_Y \cdots \times_Y X$ (the m-fiber product) by the symmetric group S_m. If a group G acts on a variety X, denote by X^{ss} the set of GIT-semistable points. If $p, q \in \mathbb{P}^2$, $p \neq q$, we set $\overline{pq} = \text{span}(p, q)$. For a positive integer g, set $m_g = 2g + 2$ and $h_g = \binom{m_g}{2}$.

2. On some properties of conic twisters

A *twister* of a curve Y is a $T \in \text{Pic}(Y)$ such that there exists a smoothing $\mathcal{Y}$ of Y such that $T \simeq \mathcal{O}_{\mathcal{Y}}(D) \otimes \mathcal{O}_Y$, where D is a Cartier divisor of $\mathcal{Y}$ supported on irreducible components of Y. If Y is of compact type, it is well-known that a twister depends only on its multidegree. A *conic twister* is a twister whose degrees on the irreducible components are positive and sum up to 2.

Proposition 2.1. *Let Y be a genus zero curve and T a conic twister of Y. Then:*

(i) *if $d_1, \dots, d_N$ are positive integers summing up to 2, then there exists a conic twister T of Y such that $\underline{\deg}\,(\omega_Y^\vee \otimes T) = (d_1, \dots, d_N)$;*

(ii) *the linear system $|\omega_Y^\vee \otimes T|$ is base point free, two-dimensional and induces a morphism $Y \to \mathbb{P}^2$ realizing Y as plane conic.*

Proof. (i) Given two components $Y_1, Y_2 \subset Y$ such that $Y_1 \cap Y_2 = \{p_1 \sim p_2\}$, the class $[p_1] - [p_2]$ is a twister T such that $T|_{Y_1} = \mathcal{O}_{Y_1}(1)$, $T|_{Y_2} = \mathcal{O}_{Y_2}(-1)$ and T is trivial on the other components of Y; the claim follows from the connectivity of Y.

(ii) If $\deg_{Y_1} T = 2$ for some $Y_1 \subset Y$, then $|\mathcal{O}_{Y_1}(T|_{Y_1})| = |\mathcal{O}_{\mathbb{P}^1}(2)| \simeq \mathbb{P}^2$, and the map $Y_1 \to \mathbb{P}^2$ is degree 2. This map extends to Y because the dual graph of the components of Y is a tree. If $\deg_{Y_1} T = \deg_{Y_2} T = 1$ for some $Y_1, Y_2 \subset Y$, let Γ be the unique path in the dual graph of the components of Y connecting Y_1, Y_2, and let $p_1 \in Y_1$, $p_2 \in Y_2$ be the unique points in Y_1, Y_2 which sit on the dual of Γ in Y. Note that T induces isomorphisms $\pi_i\colon Y_i \xrightarrow{\simeq} |\mathcal{O}_{Y_i}(T|_{Y_i})| \simeq \mathbb{P}^1$ for $i = 1, 2$. Since the dual graph of the components of Y is a tree, there is a unique map $\pi\colon Y \to \mathbb{P}^1 \cup_{\pi_1(p_1) \sim \pi_2(p_2)} \mathbb{P}^1$ extending π_1, π_2. Since up to projective morphisms there is a unique embedding $\mathbb{P}^1 \cup_{\pi_1(p_1) \sim \pi_2(p_2)} \mathbb{P}^1 \to \mathbb{P}^2$, we are done. $\square$

3. The first map

Let $\mathcal{C} \to \mathbb{P}^5 \simeq |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the universal plane conic. For any integer $m \geq 2$, consider the variety $\mathrm{Sym}^m_{\mathbb{P}^5}\mathcal{C}$ and the morphism $\rho : \mathrm{Sym}^m_{\mathbb{P}^5}\mathcal{C} \to \mathbb{P}^5$. If $k \in \mathrm{Sym}^m_{\mathbb{P}^5}\mathcal{C}$, let $\mathrm{supp}(k)$ be the conic parametrized by $\rho(k)$. The points of k are called *markings* and $(k, \mathrm{supp}(k))$, a *conic with markings*. A marking has a *weight*, i.e. the number of times it appears in k. We call the markings p_{min} and p_{max} with minimal and maximal weight, the *minimal* and *maximal markings* of k. Recall that $m_g = 2g+2$ and $h_g = \binom{m_g}{2}$, where $g \geq 3$. Set $\mathbb{P}_{h_g} = \mathrm{Sym}^{h_g}(\mathbb{P}^2)^\vee$. Consider the rational map:

$$\psi : \mathrm{Sym}^{m_g}_{\mathbb{P}^5}\mathcal{C} \dashrightarrow \mathbb{P}_{h_g} \tag{3.1}$$

where, if k has markings $\{p_i\}_{1\leq i\leq s}$ of weight $m_i = 1$ and $\{p_i\}_{s<i\leq r}$ of weights $m_i > 1$, then:

$$\psi(k) = (\cdots \underbrace{\overline{p_ip_j}, \ldots \overline{p_ip_j}}_{m_i \cdot m_j \text{ times}}, \cdots \underbrace{\mathbb{T}_{p_h}\mathrm{supp}\, k, \cdots \mathbb{T}_{p_h}\mathrm{supp}\, k}_{\binom{m_h}{2} \text{ times}}, \cdots)_{\substack{1\leq i<j\leq r \\ s<h\leq r}}. \tag{3.2}$$

Let Γ_ψ be the closure in $\mathrm{Sym}^{m_g}_{\mathbb{P}^5}\mathcal{C} \times \mathbb{P}_{h_g}$ of the graph of ψ and $p \colon \Gamma_\psi \to \mathbb{P}_{h_g}$ be the second projection. Consider the GIT quotient:

$$q : p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g} \longrightarrow \overline{J_g} = (p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g})/SL(3).$$

We say that k is *degenerate* if supp(k) is not integral. Consider the open subset V of $p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g}$ defined as:

$$V = \{r \in p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g} : r \neq p(k, r)\ \forall\ k \text{ degenerate}\} \subset p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g}$$

Recall the Hilbert-Mumford criterion [MFK, Proposition 4.3] for configurations of plane lines. Let r be in $\mathbb{P}_{h_g}$. For a point $p \in \mathbb{P}^2$, let $\mu_p(r)$ be the number of lines of r, with multiplicities, containing p. Let $\mu_l(r)$ be the multiplicity of a line l of r. Then r is GIT-semistable iff $\max_{p\in\mathbb{P}^2} \mu_p(r) \leq 2h_g/3$ and $\max_{l\in(\mathbb{P}^2)^\vee}\mu_l(r) \leq h_g/3$.

Lemma 3.1. *Let $(k, r) \in \Gamma_\psi$. Then:*

(i) *if k has a marking q of weight at least $g+1$, then r is not GIT-semistable;*
(ii) *if supp(k) is reducible and the set of markings on smooth points of a component is one marking of weight 1, then r is not GIT-semistable;*
(iii) *if supp(k) is integral and the markings have weight 1, then $r \in V$.*

Proof. (i) We have $\max_{p\in\mathbb{P}^2} \mu_p(r) \geq \mu_q(r) \geq \binom{g+1}{2} + (g+1)^2 > 2h_g/3$.

(ii) From (i), we can assume that the node n of supp(k) is a marking of weight at most g. The number of lines of r not containing n is at most $2g+1$ and hence $\max_{p\in\mathbb{P}^2} \mu_p(r) \geq \mu_n(r) \geq h_g - 2g - 1 > 2h_g/3$.

(iii) We have that r is GIT-semistable, because $\max_{p\in\mathbb{P}^2}\mu_p(r) = 2g+1 < 2h_g/3$ and $\max_{l\in(\mathbb{P}^2)^\vee}\mu_l(r) = 1 < h_g/3$. The property $\max_{l\in(\mathbb{P}^2)^\vee}\mu_l(r) = 1$ characterizes the configurations of integral conics with markings of weight 1, hence $r \in V$. □

Lemma 3.2. *Consider the rational map $\psi : Sym^{m_g}_{\mathbb{P}^5}\mathcal{C} \dashrightarrow \mathbb{P}_{h_g}$. Then the restricted morphism $\psi : \psi^{-1}(V) \to V$ is injective for every $g \geq 3$.*

Proof. Pick $k \in \mathrm{Sym}^m_{\mathbb{P}^5}\mathcal{C}$, where $m \geq 2$. Let $C = \mathrm{supp}\, k$ be integral. Set $r = \psi(k)$, as in (3.2). Let $\{m_1, \ldots, m_r\}$ be the set of the weights of k, where $m_i \leq m_{i+1}$.

Step 1. Assume that $\{m_1, \ldots, m_r\} \neq \{1, 1\}$. The goal of the first step is to recover the maximal markings of k and their weights. We claim that the maximal markings of k are the points $p \in \mathbb{P}^2$ with maximum multiplicity $\mu_p(r)$. It is easy if $m_i = 1$, for $1 \leq i \leq r$, thus assume that $m_{\max} = \max\ \{m_i\}_{1\leq i\leq r} \geq 2$. If $p \in C$, then $\mu_p(r) \leq \mu_{p_{\max}}(r)$, with the equality iff p is a maximal marking of k and we are done. If $p \notin C$, take two markings $p_i, p_j \in C$ of k of weights m_i and m_j such that $p \in \overline{p_i p_j}$ if $p_i \neq p_j$ and $p \in \mathbb{T}_{p_i}C$ if $p_i = p_j$. Thus:

i) if $p_i \neq p_j$ and $p_i, p_j \neq p_{\max}$, then $\mu_{\overline{p_i p_j}}(r) = m_i m_j < m_{\max}(m_i + m_j) = \mu_{\overline{p_i p_{\max}}}(r) + \mu_{\overline{p_j p_{\max}}}(r)$;
ii) if $p_j = p_{\max}$ and $p_i \neq p_{\max}$, then $\mu_{\overline{p_i p_{\max}}}(r) < \mu_{\overline{p_i p_{\max}}}(r) + \mu_{\mathbb{T}_{p_{\max}}C}(r)$;
iii) if $p_i = p_j \neq p_{\max}$, then $\mu_{\mathbb{T}_{p_i}C}(r) = \binom{m_i}{2} < m_i m_{\max} = \mu_{\overline{p_i p_{\max}}}(r)$;
iv) two lines $\overline{p_i p_{\max}}$ in two different cases among *i) ii) iii)* cannot be the same;
v) if $p_i = p_j = p_{\max}$, then $\mathbb{T}_{p_{\max}}C$ contains both $p_{\max}$ and p. The case ii) does not hold. If $\mathbb{T}_{p_{\max}}C$ is the only line of r containing p, then $\mu_p(r) = \mu_{\mathbb{T}_{p_{\max}}C}(r) < \mu_{\mathbb{T}_{p_{\max}}C}(r) + \mu_{\overline{p_h p_{\max}}}(r) \leq \mu_{p_{\max}}(r)$ for some $C \ni p_h \neq p_{\max}$. If p is in at least 2 lines of r, then at least one case i) or iii) holds.

This shows that $\mu_p(r) < \mu_{p_{max}}(r)$. Thus, we recover the maximal markings of k as the points of $\mathbb{P}^2$ with maximum multiplicity. In particular, we find also the number N of maximal markings of k. To recover their weight m_{max}, consider $m = \sum_{1\leq i\leq r} m_i$ and the subconfiguration r' of r of the lines containing no maximal markings of k. If $r' \neq \emptyset$, then r' is the configuration of lines associated to (k', C), where k' are the non-maximal weights of k. We know the sum of the multiplicities of the lines of r', thus we know also the sum m' of the weights of k' and $m_{\max} = (m - m')/N$. If $r' = \emptyset$, then either $m_i = m_j$ for $1 \leq i, j \leq r$ and $m_{\max} = m/N$, or $m_1 = 1 < m_i = m_j$ for $1 < i, j \leq r$ and $m_{\max} = (m-1)/N$.

Step 2. Pick $k \in \psi^{-1}(V)$ for $m = 2g+2$. We recover the markings of k, with the exception of the marking of multiplicity m_1, if $m_1 = 1 < m_2$, and of multiplicity m_1 and m_2, if $m_1 = m_2 = 1 < m_3$. In fact, using Step 1 we find the maximal markings of k and their weights. Now, consider (k', C), where k' are the non-maximal markings of k. We find the maximal markings of k' and their weights, using Step 1. By iterating, we find the configuration r_0 associated to the markings of k with minimal weights. If either $m_1 = m_2 \neq 1$ or $m_1 = m_2 = m_3 = 1$, we find the markings of k and their weights. Otherwise, let $\{p_1, \ldots, p_s\}$ be the set of the recovered markings, where $s \geq 2$, by Lemma 3.1 (i). Consider the subconfiguration r'' of r obtained by getting rid of the lines containing $p_3, \ldots, p_s$ and $\overline{p_1 p_2}$. We have three cases. In the first case, $r_0 = \emptyset$ and $m_1 = 1 < m_2$. In the second case, r_0 is a line l of multiplicity $\mu_l(r_0) > 1$ and $3 \leq m_1 < m_2$. The marking with weight m_1 is the point contained in 3 lines of r''. We recover also its weight m_1, because $\mu_l(r_0) = \binom{m_1}{2}$. In the third case, r_0 is a line l of multiplicity $\mu_l(r_0) = 1$ and either $m_1 = 2 < m_2$ or $m_1 = m_2 = 1 < m_3$. We have $m_1 = 2 < m_2$ iff r'' has 5 lines. The marking with weight 2 is the unique point contained in 3 lines of r''.

Step 3. Pick $k \in \psi^{-1}(V)$ for $m = 2g+2$. If, using Step 2, we find at least 5 markings, we recover (k, C), because we find also the markings with multiplicity 1 as the points on C with multiplicity $2g+1$. Assume that, using Step 2, we find at most 4 markings and their weights sum up to $2g+2$, i.e. they are all the markings of k. They are at least 3 markings of weights at least 2, by Lemma 3.1 (i) and Step 2. Their tangents are the lines which do not contain a pair of markings. We find at least 3 markings and 3 tangents to the markings, then we recover (k, C). If, using Step 2, we find at most 4 markings of k and their weights do not sum up to $2g+2$, then either $m_1 = 1 < m_2$ or $m_1 = m_2 = 1 < m_3$. There are 5 cases. Notice that k has at least 4 markings, by Lemma 3.1 (i).

a) 2 recovered markings and the weights are $\{1, 1, m_3, m_4\}$, $2 \le m_3 \le m_4$.

Let p_3, p_4 be the markings with multiplicities m_3, m_4. If $m_3 \ne 3$, then $\mathbb{T}_{p_3}C$ is the line through p_3 not containing p_4 and whose multiplicity is not m_3. Similarly, we determine also $\mathbb{T}_{p_4}C$. Consider the 4 lines of the configuration r different from $\mathbb{T}_{p_3}C, \mathbb{T}_{p_4}C, \overline{p_3p_4}$ and containing either p_3 or p_4. The pairwise intersections of these lines are 6 points: two points of multiplicity $2g+1$, two of multiplicity $2g$ and p_3, p_4. The points of multiplicity $2g+1$ are the markings with weight 1. Thus we recover 4 markings and 2 tangents to the markings, hence also $(k, \mathrm{supp}\ k)$. If $m_3 = 3$, then also $m_4 = 3$ from Lemma 3.1 (i) and $g = 3$. It is easy to see that there is only one conic with markings having r as associated configuration.

b) 3 recovered markings and the weights are $\{1, m_2, m_3, m_4\}$, $2 \le m_2 \le m_3 \le m_4$.

Let p_i have weight m_i. Assume that two weights are not equal to 3, e.g. $m_2, m_3 \ne 3$. Consider the subconfiguration r' of r of the lines not containing p_4 and different from $\overline{p_2p_3}$. The marking with weight 1 is the point $p \ne p_2, p_3$ with multiplicity $\mu_p(r') = m_2 + m_3$. If two weights are equal to 3, then $m_2 = m_3 = 3$, $m_4 = 2g - 5$. If $m_4 \ne 3$, then $\mathbb{T}_{p_4}C$ is the unique line containing p_4 with multiplicity $\binom{m_4}{2}$ and different from $\overline{p_2p_4}, \overline{p_3p_4}$. Consider the subconfiguration r' of r obtained by getting rid of $\mathbb{T}_{p_4}C$. Then r' is the configuration of (k', C), where k' are the markings of k with set of weights $\{1, 1, 3, 3\}$, where p_4 has weight 1 in k'. Using (a), we recover (k', C), hence also the original (k, C). If $m_2 = m_3 = m_4 = 3$, it is easy to see that there is only one conic with markings having r as associated configuration.

c) 3 recovered markings and the weights are $\{1, 1, m_3, m_4, m_5\}$, $2 \le m_3 \le m_4 \le m_5$.

There is a marking of weight different form 3, otherwise $2g + 2 = 11$, for example $m_3 \ne 3$. Let r' be the subconfiguration of r obtained by getting rid of lines containing the points with weights m_4, m_5. Thus the markings with weight 1 are the points p, q of multiplicity $\mu_p(r') = \mu_q(r') = m_3 + 1$. We recover (k, C).

d) 4 recovered markings and the weights are $\{1, m_2, m_3, m_4, m_5\}$, $2 \le m_2 \cdots \le m_5$.

Consider the subconfiguration r' of r obtained by getting rid of the lines connecting any two points with multiplicity > 1. Then the point with multiplicity 1 is the only point contained in 4 lines of r'. We recover (k, C).

e) 4 recovered markings and the weights are $\{1, 1, m_3, \dots, m_6\}$, $2 \le m_3 \cdots \le m_6$.

If a marking has a weight different form 3, we argue as in c). Otherwise, consider the subconfiguration r' of r obtained by getting rid of the lines containing the markings with weight m_5, m_6. Thus r' is the configuration of (k', C), where k' has weights $\{1, 1, 3, 3\}$ and we argue as in a). □

Denote by J_g the subset of $\overline{J_g}$ corresponding to the classes of configurations of lines of integral conics with markings of weight 1.

Theorem 3.3. *There exists a map $\alpha_g \colon \overline{J_g} \dashrightarrow \overline{B_{m_g}}$ defined at least over J_g.*

Proof. Up to restrict V, we have that $\psi : \psi^{-1}(V) \to V$ is an isomorphism by Theorem 3.2. Pick $k \in \psi^{-1}(V)$. Since $\mathrm{supp}(k)$ is irreducible, Lemma 3.1 (i) implies that $(k, \mathrm{supp}(k))$ is a GIT-stable binary form. Thus $\mathcal{U} \to \psi^{-1}(V) \simeq V$ is a family of stable binary forms, where $\mathcal{U} = \{\underline{(k, p)} : p \in \mathrm{supp}\ k\} \subset \psi^{-1}(V) \times \mathbb{P}^2$, hence we get a $SL(3)$-invariant morphism $V \to \overline{B_{m_g}}$, inducing the rational map $\alpha_g \colon \overline{J_g} \dashrightarrow \overline{B_{m_g}}$.

To show that α_g is defined over J_g, it is enough to show that the differential of ψ is injective over an irreducible conics with markings of weight 1. We show that, if U is the open subset of $\mathrm{Sym}^{m_g}(\mathbb{P}^2)$ of m_g distinct points such that any three of them are not contained in a line, then the differential of $\psi\colon U \to \mathbb{P}_{h_g}$ is injective, where $\psi(p_1,\dots p_{m_g}) = (\dots \overline{p_ip_j}\dots)_{1\le i<j\le m_g}$. If $k\in U$, set $X=\psi(k)$, the union of h_g distinct lines. If $\mathcal{N}_{X/\mathbb{P}^2}$ is the normal sheaf of X in $\mathbb{P}^2$, then:

$$\mathbb{T}_kU = \mathbb{T}_{p_1}\mathbb{P}^2\oplus\cdots\oplus\mathbb{T}_{p_{m_g}}\mathbb{P}^2 \xrightarrow{d\psi} H^0(X,\mathcal{N}_{X/\mathbb{P}^2}).$$

For $v_i\in\mathbb{T}_{p_i}\mathbb{P}^2$, let $d\psi(v_1,\dots v_{m_g})=0$, i.e. it is the trivial embedded deformation of X, fixing all the components of X. This means that v_i is contained in the lines of X containing p_i. It is impossible, being p_i contained in at least two lines of X. □

4. The second map and the factorization

A family of m-pointed stable curves of genus zero is a family $f:\mathcal{Y}\to B$ of curves of genus zero with sections $\sigma_1,\dots,\sigma_m$ of f such that $(Y_b,\sigma_1(b),\dots,\sigma_m(b))$ is a m-pointed stable curve of genus zero for $b\in B$. If T is a conic twister of Y, let $\varphi_T: Y\to\mathbb{P}^2$ be the morphism induced by $|\omega_Y^\vee\otimes T|$ as in Lemma 2.1 (ii).

Definition 4.1. Let $(Y,p_1,\dots,p_m)$ be a m-pointed stable curve of genus zero. A connected subcurve $P\subset Y$ is *a principal part* if there exists a conic twister T of Y such that the point $k=(\varphi_T(p_1),\dots,\varphi_T(p_m)\in \mathrm{Sym}^m_{\mathbb{P}^5}\mathcal{C}$ satisfies $\mathrm{supp}\, k=\varphi_T(P)=\varphi_T(Y)$ and $\psi(k)\in\mathbb{P}^{ss}_{h_g}$, where $\psi:\mathrm{Sym}^{m_g}_{\mathbb{P}^5}\mathcal{C}\dashrightarrow\mathbb{P}_{h_g}$ is the map (3.1).

A principal part P has at most two components. If P is a principal part of $(Y,p_1,\dots,p_{m_g})$, the associated conic twister T is uniquely determined by the condition $\varphi_T(P)=\varphi_T(Y)$. Since P is connected, $\varphi_T(p_i)$ is not in the singular locus of $\varphi_T(P)$ and hence the map ψ is defined over $k=(\varphi_T(p_1),\dots,\varphi_T(p_m))$.

Example 4.2. Let $(Y,p_1,\dots,p_{10})$ be a pointed stable curve where $Y=Y_1\cup Y_2$. Assume that $p_1,\dots,p_6\in Y_1$ and $p_7,\dots,p_{10}\in Y_2$. Both Y and Y_1 are principal parts. In fact, if we consider $k_i=(\varphi_{T_i}(p_1),\dots,\varphi_{T_i}(p_{10}))$, where $T_1=\mathcal{O}_Y$ and T_2 is the twister given by Y_2, then $\psi(k_1)$ and $\psi(k_2)$ are GIT-semistable.

We refer to [K] for a proof of the following Lemma.

Lemma 4.3. *Let $[f:\mathcal{Y}\to B,\sigma_1,\dots,\sigma_m]$ be a family of m-pointed stable curves of genus zero. Then there exists a unique family $[f':\mathcal{Y}'\to B,\sigma_1',\dots,\sigma_{m-1}']$ and a B-morphism $h\colon\mathcal{Y}\to\mathcal{Y}'$ such that $h\circ\sigma_i=\sigma_i'$ for $i=1,\dots,m-1$. If $E_b\subset f^{-1}(b)$ is the component with $\sigma_m(b)\in E_b$ and $h_b=h|_{f^{-1}(b)}:f^{-1}(b)\to(f')^{-1}(b)$, then:*

(i) *h_b contracts E_b iff $\Big|E_b\cap\overline{(f^{-1}(b)-E_b)})|+|E_b\cap\{\sigma_1(b),\dots,\sigma_{m-1}(b)\}\Big|\le 2$;*
(ii) *if h_b does not contract E_b, then h_b is an isomorphism;*
(iii) *if h_b contracts E_b, then $h_b|_{\overline{f^{-1}(b)-E_b}}$ is an isomorphism.*

Lemma 4.4. *The subset of $\overline{M_{0,m_g}}$ of the curves with a principal part is an open subset containing the locus of the curves with at most two components.*

Proof. Let P be a principal part of $(Y,p_1,\dots,p_{m_g})$. Let $[f:\mathcal{Y}\to B,\sigma_1,\dots,\sigma_{m_g}]$ be a family of m_g-pointed stable curves of genus zero with $Y=f^{-1}(0)$ and $p_i=\sigma_i(0)$ for $0\in B$. Applying Lemma 4.3, we get a family $f':\mathcal{Y}'\to B$ with $P=(f')^{-1}(0)$ and a morphism $h\colon\mathcal{Y}\to\mathcal{Y}'$. Now, P has at most 2 components, then up to shrinking B to an open subset containing 0, all the fibers of $\mathcal{Y}'$ have at most two components.

The image of the map $\varphi : \mathcal{Y}' \to \mathbb{P}(H^0(f'_*\omega_{f'}^\vee)^\vee)$ induced by $|\omega_{f'}^\vee|$ is a family of conics over B. The conic over $b \in B$ has markings $\gamma(b) = (\gamma_1(b), \dots, \gamma_{m_g}(b))$, where $\gamma_i = \varphi \circ h \circ \sigma_i$. By construction a marking is not a node of a fiber, hence $\psi : \mathrm{Sym}^{m_g}_{\mathbb{P}^5}\mathcal{C} \dashrightarrow \mathbb{P}_{h_g}$ is defined over $\gamma(b)$. Now, $\psi \circ \gamma(0) \in \mathbb{P}^{ss}_{h_g}$ because P is a principal part. Thus, up to shrinking again B, we have $\psi \circ \gamma(b) \in \mathbb{P}^{ss}_{h_g}$ for $b \in B$ and the fiber of $\mathcal{Y}' \to B$ over b is a principal part of $f^{-1}(b)$ and we are done.

We show that Y has a principal part if it has at most two components. If Y is irreducible and $T = \mathcal{O}_Y$, then $\psi(\varphi_T(p_1), \dots, \varphi_T(p_{m_g}))$ is GIT-stable by Lemma 3.1 (iii), thus Y is a principal part. If $Y = Y_1 \cup Y_2$, let $[f : \mathcal{Y} \to B, \sigma_1, \dots, \sigma_{m_g}]$ be a general smoothing of $(Y, p_1, \dots, p_{m_g})$ and $\mathcal{C}^* \to B^*$ be the family of conics given by $|(\omega_f|_{f^{-1}(B^*)})^\vee|$. We have a map $\varphi\colon \mathcal{Y} \dashrightarrow \mathcal{C}^*$, which is an isomorphism away from Y. Now, $\psi(\varphi \circ \sigma_1(b), \dots, \varphi \circ \sigma_{m_g}(b)) \in \mathbb{P}^{ss}_{h_g}$ for $b \in B^*$ by Lemma 3.1 (iii). By the GIT-semistable replacement property, up to a finite base change totally ramified over $0 \in B$, we find a completion $f'\colon \mathcal{C} \to B$ and $k \in \mathrm{Sym}^{m_g}(C)$, where $C = (f')^{-1}(0)$ such that $\psi(k) \in \mathbb{P}^{ss}_{h_g}$. If φ induces a morphism $\varphi\colon \mathcal{Y} \to \mathcal{C}$, then $\varphi|_Y = \varphi_T$, where either $T = \mathcal{O}_Y$ or T is given by Y_1 or Y_2 and Y has a principal part. Otherwise, call $\widetilde{\varphi} : \widetilde{\mathcal{Y}} \to \mathcal{C}$ the regularization of φ. Let φ be not defined at $p \in Y$ and E be an exceptional components over p. Let p be a smooth point of Y. Then $\widetilde{\varphi}(E) \subset C$ is a component containing at most one marking with weight 1 and $\psi(k) \notin \mathbb{P}^{ss}_{h_g}$ by Lemma 3.1 (ii), a contradiction. Assume that $p_1, \dots, p_t \in Y_1$ for $t \leq m_g/2$. Let p be the node of Y. If $\widetilde{\varphi}$ contracts Y_2, then k has a marking with weight at least $m_g/2$ and $\psi(k) \notin \mathbb{P}^{ss}_{h_g}$ by Lemma 3.1 (ii), a contradiction. If Y_2 is not contracted, then $C = \widetilde{\varphi}(Y_2 \cup E)$. If $q = \widetilde{\varphi}(p_1) = \dots = \widetilde{\varphi}(p_t)$, then $k = (q, \widetilde{\varphi}(p_{t+1}), \dots, \widetilde{\varphi}(p_{m_g}))$ and $\psi(k) \in \mathbb{P}^{ss}_{h_g}$. Consider $T = \mathcal{O}_T$ and $k' = (\varphi_T(p_1), \dots, \varphi_T(p_{m_g}))$. We have $\psi(k') \in \mathbb{P}^{ss}_{h_g}$ because $\max_{p \in \mathbb{P}^2} \mu_p(\psi(k')) \leq \max_{p \in \mathbb{P}^2} \mu_p(\psi(k))$ and $\max_{l \in (\mathbb{P}^2)^\vee} \mu_l(\psi(k')) \leq \max_{l \in (\mathbb{P}^2)^\vee} \mu_l(\psi(k))$, hence Y is a principal part. $\square$

Let $P_g \subset \overline{M_{0,m_g}}$ be the open subset of the curves with a principal part and $\overline{N_{0,m_g}} = \overline{M_{0,m_g}}/S_{m_g}$ the moduli space of m_g-marked stable curves, where S_{m_g} is the symmetric group.

Theorem 4.5. *There exists a map $\beta_g\colon \overline{N_{0,m_g}} \dashrightarrow \overline{J_g}$ defined at least over P_g/S_{m_g}.*

Proof. First of all, we show that if $(Y, p_1, \dots, p_{m_g}) \in \overline{M_{0,m_g}}$ has two principal parts P_1 and P_2, whose associated configurations of lines are r_1 and r_2, then:

$$\overline{O_{SL(3)}(r_1)} \cap \overline{O_{SL(3)}(r_2)} \cap \mathbb{P}^{ss}_{h_g} \neq \emptyset, \tag{4.3}$$

where $O_{SL(3)}(\cdot)$ denotes the orbit under the action of $SL(3)$. In fact, consider a general smoothing $[f : \mathcal{Y} \to B, \sigma_1, \dots, \sigma_{m_g}]$ of $(Y, p_1, \dots, p_{m_g})$. For $j = 1, 2$, let T_j be the twister of P_j. The morphisms φ_{T_j} induced by $|\omega_f^\vee \otimes T_j|$ give rise to two families of conics, which are isomorphic away from the special fiber. Furthermore, $\{\varphi_{T_1} \circ \sigma_i\}$ and $\{\varphi_{T_2} \circ \sigma_i\}$ induce markings on the two families. By construction, the associated families of configurations of lines are $SL(3)$-conjugate over B^* and r_1 and r_2 are their special fibers. Thus, r_1 and r_2 are GIT-semistable limits of conjugate families of GIT-stable configurations, hence (4.3) follows.

Now, let $\mathcal{P}_g$ be the functor of familes of m_g-pointed stable curves with a principal part. We construct a functor tranformation $\mathcal{P}_g \to \mathcal{M}or(-, \overline{J_g})$. For a scheme B, pick $[f : \mathcal{Y} \to B, \sigma_1, \dots, \sigma_{m_g}] \in \mathcal{P}_g(B)$. As in the proof of Lemma 4.4 (ii), we get an open covering $B = \cup B_h$ of B and morphisms $t_h\colon B_h \to \mathbb{P}^{ss}_{h_g}$ such that $t_h(b)$ is the configuration of lines associated to a principal part of Y_b for $b \in B_h$. Consider the quotient morphism $q : p(\Gamma_\psi) \cap \mathbb{P}^{ss}_{h_g} \to \overline{J_g}$. Now, $t_h(B_h \cap B_k)$ and $t_k(B_h \cap B_k)$ are congruent (in an algebraic way) modulo $SL(3)$, i.e. we can glue the morphisms $q \circ t_h$ to a morphism $B \to \overline{J_g}$. By (4.3), this morphism does not depend on the principal part of Y_b, hence we get the required functor transformation. Now, P_g coarsely represents $\mathcal{P}_g$, thus we get a morphism $P_g \to \overline{J_g}$ which is S_{m_g}-invariant by construction, hence we get also a morphism $\beta\colon P_g/S_{m_g} \to \overline{J_g}$. □

Let us recall how the morphism $F_g\colon \overline{N_{0,m_g}} \to \overline{B_{m_g}}$ is defined in [AL]. Let $(Y, p_1, \dots, p_{m_g}) \in \overline{N_{0,m_g}}$. The *weighted dual tree* of Y is the dual graph Γ_Y of Y and, for each vertex, the number of marked points contained in the corresponding component. If Γ is a subset of Γ_Y, let $wt(\Gamma)$ be the sum of the weights of the vertices contained in Γ. We say that Y has a *central vertex* $v \in \Gamma_Y$ if $wt(\Gamma) < m_g/2$ for every connected subsets Γ of $\Gamma_Y \backslash v$. The following is [AL, Lemma 3.2].

Lemma 4.6. *Let $(Y, p_1, \dots, p_{m_g}) \in \overline{N_{0,m_g}}$. Then Y has a central vertex if and only if there are no edges e of Γ such that $wt(\Gamma_1) = wt(\Gamma_2) = m_g/2$, where Γ_1, Γ_2 are the connected subgraphs such that $\Gamma_1 \cup \Gamma_2 = \overline{\Gamma - e}$. There exists at most one central vertex.*

A marked stable curve has a central vertex if it is not contained in the divisor Δ of $\overline{N_{0,m_g}}$ whose general point has two components containing $g + 1$ marked points. If the central vertex exists, then $F_g(Y, p_1, \dots p_{2g+2})$ is obtained by contracting all the components of Y, which do not correspond to the central vertex. If Y has no central vertex, it is easy to see that the edge disconnecting Γ_Y in two subgraphs with weights $g + 1$ is unique. We call it *the central edge* of Γ.

Theorem 4.7. *The variety $\overline{J_g}$ is a compactification of H_g and the chain of maps $\overline{N_{0,m_g}} \overset{\beta_g}{\dashrightarrow} \overline{J_g} \overset{\alpha_g}{\dashrightarrow} \overline{B_{m_g}}$ gives a rational factorization of $F_g : \overline{N_{0,m_g}} \to \overline{B_{m_g}}$.*

Proof. The morphism β_g restricts to an injection $H_g \to \overline{J_g}$ whose image is the subset J_g of Theorem 3.3. The inverse is the morphism of Theorem 3.3.

To prove the factorization, it is enough to show that, if P is an irreducible principal part of a pointed stable curve $(Y, p_1, \dots, p_{m_g})$, then P is the component corresponding to a central vertex of Γ_Y. First of all, assume that Y has a (unique) central vertex v and let $C \subset Y$ be the corresponding component. Assume that $P \neq C$ and let v_P be the vertex of P. There is an edge e of Γ_Y such that, if Γ_1, Γ_2 are the connected subgraphs with $\Gamma_1 \cup \Gamma_2 = \overline{\Gamma_Y - e}$, then $v_P \in \Gamma_1$ and $v \in \Gamma_2$. Thus $wt(\Gamma_1) < m_g/2$ by definition of central vertex. Let T be the conic twister such that $\omega_Y^\vee \otimes T$ has degree 2 on P. Now, $\psi(\varphi_T(p_1), \dots, \varphi_T(p_{m_g})) \in \mathbb{P}^{ss}_{h_g}$, by definition of principal part. Since φ_T contracts the connected subcurve of Y corresponding to Γ_2 to a unique marking, we have $wt(\Gamma_2) < m_g/2$ by Lemma 3.1 (i). Then $m_g = wt(\Gamma) = wt(\Gamma_1) + wt(\Gamma_2) < m_g$, a contradiction. If Y has no central edge, let e be the central edge of Y. Set $\Gamma_1 \cup \Gamma_2 = \overline{\Gamma_Y - e}$ for connected graphs Γ_1, Γ_2. If T is the twister of P, then φ_T contracts either the component of Γ_1 or of Γ_2. Now, $\psi(\varphi_T(p_1), \dots, \varphi_T(p_{m_g})) \in \mathbb{P}^{ss}_{h_g}$ and $(\varphi_T(p_1), \dots, \varphi_T(p_{m_g}))$ has a marking of weight at least $g + 1$, because $wt(\Gamma_1) = wt(\Gamma_2) = g + 1$, contradicting Lemma 3.1 (i). □

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References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves.* Vol. I. Fund. Princ. of Math. Sciences. vol 267, Springer-Verlag, New York, 1985.

[AL] D. Avritzer, H. Lange. *The moduli space of hyperelliptic curves and binary forms.* Math. Z., vol. 242, 2002, n.4, 615-632.

[CCC] L. Caporaso, C. Casagrande, M. Cornalba, *Moduli of roots of line bundles on curves.* Trans. of the Amer. Math. Soc., to appear.

[CS1] L. Caporaso, E. Sernesi, *Recovering plane curves from their bitangents.* J. Alg. Geom. vol. 12, 2003, n. 2, 225–244.

[CS2] L. Caporaso, E. Sernesi, *Characterizing curves by their odd theta-characteristics*. J. R. Angew. Math. **562** 2003, 101–135.

[GS] S. Grushevsky, R. Salvati Manni, *Gradients of odd theta functions*. J. R. angew. Math. **573** 2004, 43–59.

[H] B. Hassett, *Moduli spaces of weighted pointed stable curves.* Ad. Math, 173(2):316-352, 2003.

[K] F. Knudsen, *Projectivity of the moduli space of stable curves, II.* Math. Scand., vol. 52, 1983, 1225-1265.

[L] D. Lehavi, *Any smooth quartic can be reconstructed from its bitangents.* Isr. J. Math. 146, 2005, 371–379.

[M] D. Mumford, *Tata lectures on theta II.* Progress in math., 43, Birkhäuser Boston Inc. Boston, 1984.

[MFK] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory.* E.M.G. 34. Third edition, Springer, New York, 1994.

Dipartimento di Matematica Guido Castelnuovo, Università Roma La Sapienza, piazzale Aldo Moro 2, 00185 Roma, Italia

Instituto de Matematica Pura e Aplicada, Estrada D. Castorina 110, 22460-320 Rio de Janeiro, Brazil

E-mail address: `pacini@impa.br`