ENRICHED SPIN CURVES ON STABLE CURVES WITH TWO COMPONENTS

MARCO PACINI

Abstract. In [M], Mainò constructed a moduli space for enriched stable curves, by blowing-up the moduli space of Deligne-Mumford stable curves. We introduce enriched spin curves, showing that a parameter space for these objects is obtained by blowing-up the moduli space of spin curves.

1. Introduction

A basic tool in the theory of limit linear series is to consider degenerations of smooth curves to singular ones. In [EH], Eisenbud and Harris developed a theory for curves of compact type, i.e. curves having only separating nodes. The advantage to work with curves of compact type is the following. Let B be the spectrum of a DVR and $f\colon \mathcal{C} \to B$ be a general smoothing of a nodal curve C, i.e. $C = f^{-1}(0)$ for some $0 \in B$ and $f^{-1}(b)$ is a smooth curve for $b \neq 0$ and $\mathcal{C}$ is smooth. If $C_1 \dots, C_\gamma$ are the components of C, then all the extensions of a line bundle $\mathcal{L}^*$ over $f^{-1}(B-0)$ are given by $\mathcal{L} \otimes \mathcal{O}_\mathcal{C}(C_i)$, where $\mathcal{L}$ is a fixed extension. If C is of compact type, then $\mathcal{L} \otimes \mathcal{O}_\mathcal{C}(C_i)$ does not depend on the smoothing. This is not true in general and it is the main difficulty arising when one tries to extend the theory to a more general class of curves. The problem was solved in [EM] for general curves with two components, but a general analysis is still not available.

With these motivations, the notion of *enriched stable curve* is introduced in [M]. Let C be a stable curve with components $C_1 \dots, C_\gamma$. An *enriched stable curve of* C is given by $(C, \mathcal{O}_\mathcal{C}(C_1)|_C, \dots, \mathcal{O}_\mathcal{C}(C_\gamma)|_C)$, where $f\colon \mathcal{C} \to B$ is a general smoothing of C. Necessarily, we have $\otimes_i^\gamma \mathcal{O}_\mathcal{C}(C_i)|_C \simeq \mathcal{O}_C$. In [M], it is shown that an enriched stable curve of C only depends on the first order deformation of the given smoothing $\mathcal{C}$ of C. Furthermore it is possible to understand when two first order deformations of C give rise to the same enriched stable curve. A moduli space for enriched stable curves is constructed by taking blow-ups of the base of the universal deformation of stable curves and glueing all these blow-ups together.

On the other hand for a given family of nodal curves $f\colon \mathcal{C} \to B$ and a line bundle $\mathcal{N}$ of $\mathcal{C}$ of relative even degree, viewed as a family of line bundles on

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the fibers of f, one can consider the problem of compactifying the moduli space for roots of the restriction of $\mathcal{N}$ to the fibers of f. In [CCC], a moduli space is constructed in terms of *limit square roots*. In particular, when $f\colon \mathcal{C} \to B$ is a stable family and $\mathcal{N} = \omega_f$, this moduli space represents *spin curves of stable curves*, a generalization of theta characteristics on smooth curves. In [C], a moduli space $\overline{S_g}$ for spin curves of stable curves of genus g is constructed. The moduli space $\overline{S_g}$ is endowed with a natural finite morphism $\varphi\colon \overline{S_g} \longrightarrow \overline{M_g}$ onto the moduli space of Deligne–Mumford stable curves. As one can expect, the degree of φ is 2^{2g}. The fibers of φ over represent *spin curves*. The paper [CC] provides an explicit combinatorial description of the boundary.

Since a parameter space for enriched curves is obtained by blowing-up $\overline{M_g}$, we expect that a point of a blow-up of $\overline{S_g}$ parametrizes roots of all the possible degenerations of the dualizing sheaf on families of stable curves. Indeed, let C be a stable curve. A curve X is obtained from C by *blowing-up* a subset Δ of the set of the nodes of C if there is a morphism $\pi\colon X \to C$ such that, for every $p_i \in \Delta$, $\pi^{-1}(p_i) = E_i \simeq \mathbb{P}^1$ and $\pi\colon X - \cup_i E_i \to C - \Delta$ is an isomorphism. The curves E_i are called *exceptional*. Let C be with two smooth components, C_1, C_2. We define an *enriched spin curve supported on* X as a tern (X, L_1, L_2), where X is a blow-up of C at a proper subset of nodes, and L_1, L_2 are line bundles of X such that:

(i) L_i has degree one on exceptional components of X;
(ii) if $\widetilde{X}$ is the complement of the union of the exceptional components of X, then $(L_i|_{\widetilde{X}})^{\otimes 2} \simeq \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(C_i)|_{\widetilde{X}}$, for $i = 1, 2$, where $\widetilde{\mathcal{X}} \to B$ is a general smoothing of X, and $(L_1)|_{\widetilde{X}} \otimes (L_2)|_{\widetilde{X}} \simeq \omega_{\widetilde{X}}$.

We introduce a notion of isomorphism between enriched spin curves and we denote by $\overline{\mathcal{SE}_C}$ the set of isomorphism classes of enriched spin curves and by $\mathcal{SE}_X$ the subset of the ones supported on X. In Lemma 2.2, we show when $\overline{S_g}$ is singular at a spin curve ξ of a curve C with two smooth components. A detailed analysis of the singular locus of $\overline{S_g}$ is given in [L]. We consider a distinguished subset D_X of $\overline{S_g}$ containing ξ as singular point and we find a blow-up $D_X^\nu \to D_X$, desingularizing D_X, with exceptional divisor $\mathbb{P}_\xi^{\delta-1}$, where δ is the number of nodes of C. The following theorem sums-up Proposition 3.4 and Theorem 3.6, 3.7.

Theorem 1.1. *Let C be with δ nodes and two smooth components of genus at least 1. Assume that $Aut(C) = \{id\}$. For ξ running over the set of spin curves of C which are singular points of $\overline{S_g}$, there exist δ hyperplanes $H_{\xi,1}, \dots, H_{\xi,\delta}$ of $\mathbb{P}_\xi^{\delta-1}$, such that:*

(i) *$\mathcal{SE}_C$ and $\cup_\xi(\mathbb{P}_\xi^{\delta} - (\cup_{1\le i\le\delta} H_{\xi,i}))$ are isomorphic torsors;*
(ii) *if X_I and $\widetilde{X_I}$ are the blow-up and the normalization of C at a subset $I = \{p_1, \dots, p_h\}$ of nodes of C, with $1 \le h < \delta$, then the set of the*

isomorphism classes of enriched spin curves of C supported on X_I, $\mathcal{SE}_{\widetilde{X_I}}$ and $\cup_\xi(\cap_{1\leq i\leq h} H_{\xi,i} - \cup_{h<i\leq\delta} H_{\xi,i})$ are isomorphic torsors.

The proof of the Theorem 1.1 uses some ideas of [P]. We see that $\overline{\mathcal{SE}_C}$ is parametrized by a complete variety and that it is stratified in terms of enriched spin curves of partial normalizations of C, as illustrated in Example 3.8. Furthermore, recall that the moduli space of enriched stable curve constructed in [M] is not complete. The analysis of this paper suggests that a compactification of this moduli space could be given in terms of enriched stable curves on partial normalizations of stable curves.

Although the hypothesis that the components of C are smooth could be removed in Theorem 1.1, the combinatorics involved became a bit harder especially in the proof of Theorem 3.7. Therefore, we choose to present the simplest case in this paper and we plan to investigate the problem of the generalization to any stable curve in a different paper.

We will use the following notation and terminology. We work over the field of complex numbers. A *curve* is a connected projective curve which is Gorenstein and reduced. A *stable* (*semistable*) curve C is a nodal curve such that every smooth rational subcurve of C meets the rest of the curve in at least 3 points (2 points). Let ω_X be the dualizing sheaf of a curve X. The genus of X is $g = h^0(X, \omega_X)$. If $Z \subset X$ is a subcurve, set $Z^c := \overline{X - Z}$. A *family of curves* is a proper and flat morphism $f\colon \mathcal{W} \to B$ whose fibers are curves. We denote either by ω_f or by $\omega_{\mathcal{W}/B}$, the relative dualizing sheaf of a family. A *smoothing* of a curve X is a family $f\colon \mathcal{X} \to B$, where B is a smooth, connected, affine curve of finite type, with a distinguished point $0 \in B$, such that $X = f^{-1}(0)$ and $f^{-1}(b)$ is smooth for $b \in B - 0$. A *general smoothing* is a smoothing with smooth total space. A curve X is obtained from C by *blowing-up* a subset Δ of the set of the nodes of C, if there is a morphism $\pi\colon X \to C$ such that, for every $p_i \in \Delta$, $\pi^{-1}(p_i) = E_i \simeq \mathbb{P}^1$ and $\pi\colon X - \cup_i E_i \to C - \Delta$ is an isomorphism. For every $p_i \in \Delta$, we call E_i an *exceptional component*. If X is a curve , we denote by $\mathrm{Aut}(X)$ the group of automorphisms of X.

2. The moduli space of spin curves

In [CCC], the authors described compactifications of moduli spaces of roots of line bundles on smooth curves, in terms of *limit square roots*.

Let C be a nodal curve and let $N \in \mathrm{Pic}(C)$ be of even degree. A tern (X, L, α), where $\pi\colon X \to C$ is a blow-up of C, L is a line bundle on X and α is a homomorphism $\alpha\colon L^{\otimes 2} \to \pi^*(N)$, is a *limit square root* of (C, N) if:

(i) the restriction of L to every exceptional component has degree 1;
(ii) the homomorphism α is an isomorphism at the points of X not belonging to an exceptional component;
(iii) for every exceptional component E such that $E \cap E^c = \{p, q\}$ the orders of vanishing of α at p and q add up to 2.

The curve X is called the *support* of the limit square root. If C is stable, then a limit square root of (C, ω_C) is said to be a *spin curve of* C.

If X is a blow-up of a nodal curve C, denote by $\widetilde{X} := \overline{X - \cup E}$, where E runs over the set of the exceptional components. There exists a notion of isomorphism of limit square roots. By [C, Lemma 2.1], two limit square roots $\xi = (X, L, \alpha)$ and $\xi' = (X, L', \alpha')$ are isomorphic if and only if the restrictions of L and L' to $\widetilde{X}$ are isomorphic. Denote by $\mathrm{Aut}(\xi)$ the group of automorphisms of ξ. A limit square root of (C, N) supported on a blow-up X with exceptional components $\{E_i\}$ is determined by the line bundle L obtained by glueing $\mathcal{O}_{E_i}(1)$, for every E_i, and a square root of $(\pi^*N)|_{\widetilde{X}}(\sum(-p_i - q_i))$, where $\{p_i, q_i\} = E_i \cap E_i^c$. Indeed, it is possible to define a homomorphism α such that (X, L, α) is a limit square root. In the sequel, if no confusion may arise, we denote a limit square root simply by (X, L). Let $f\colon \mathcal{C} \to B$ be a family of nodal curves over a quasi-projective scheme B and let $\mathcal{N} \in \mathrm{Pic}(\mathcal{C})$ be of even relative degree. There exists a quasi-projective scheme $\overline{S}_f(\mathcal{N})$, finite over B, which is a coarse moduli space, with respect to a suitable functor, of isomorphism classes of limit square roots of the restriction of $\mathcal{N}$ to the fibers of f. For more details, we refer to [CCC, Theorem 2.4.1].

Let C be a nodal curve and $N \in \mathrm{Pic}(C)$ of even degree. Denote by $\overline{S}_C(N)$ the zero-dimensional scheme $\overline{S}_{f_C}(N)$, where $f_C\colon C \to \{pt\}$ is the trivial family. In particular, $\overline{S}_C(N)$ is in bijection with the isomorphism classes of limit square roots of (C, N). If $f\colon \mathcal{C} \to B$ is a family of curves and $\mathcal{N} \in \mathrm{Pic}\,\mathcal{C}$, then the fiber of $\overline{S}_f(\mathcal{N}) \to B$ over $b \in B$ is $\overline{S}_{f^{-1}(b)}(\mathcal{N}|_{f^{-1}(b)})$, as explained in [CCC, Remark 2.4.3]. Denote by Σ_X the graph having the connected components of $\widetilde{X}$ as vertices and the exceptional components as edges. By [CCC, 4.1], the multiplicity of $\overline{S}_C(N)$ in $\xi = (X, G, \alpha)$ is $2^{b_1(\Sigma_X)}$.

In [C], the author constructed the moduli space $\overline{S_g}$ of spin curves of stable curves of genus g. The moduli space $\overline{S_g}$ is endowed with a finite map $\varphi\colon \overline{S_g} \to \overline{M_g}$, of degree 2^{2g}. Let $\overline{M_g^0}$ be the open subset parametrizing curves without non-trivial automorphisms and let $\overline{S_g^0}$ be the restriction of $\overline{S_g}$ over $\overline{M_g^0}$. In this case, if $f\colon \mathcal{C} \to B$ is a family of stable curves with moduli morphism $B \to \overline{M_g^0}$, then $\overline{S}_f(\omega_f) = \overline{S_g} \times_{\overline{M_g}} B$.

Notation 2.1. Let C be a stable curve with $\mathrm{Aut}(C) = \{id\}$ and nodes $p_1, \dots, p_\delta$. Let $\mathrm{Def}(C)$ be the base of the universal deformation of C, which is a $(3g-3)$-dimensional polydisc in $\mathbb{C}^{3g-3}_{t_1,\dots,t_{3g-3}}$. Here $\{t_i = 0\}$ is the locus where the node p_i is preserved. In particular, locally analytically at C, we have $\mathrm{Def}(C) \subset \overline{M_g}$. Denote by $D_C = \mathrm{Def}(C) \cap \mathbb{C}^{\delta}_{t_1,\dots,t_\delta}$ and by $D_X = \varphi^{-1}(D_C)$, where $\varphi\colon \overline{S_g} \to \overline{M_g}$.

Lemma 2.2. *Let C be a stable curve with two smooth components and δ nodes with $Aut(C) = \{id\}$. Let ξ be a spin curve of C. Then, $\overline{S_g}$ is singular at ξ if and only if ξ is supported on the blow-up at the whole set of nodes of*

C. In this case, locally analytically at ξ, the equations of D_X are of type:

$$w_{ii}w_{jj} = w_{ij}^2, \ \ w_{ii}w_{jj}w_{kk} = w_{ij}w_{jk}w_{ik}, \ \text{for } 1 \leq i < j < k \leq \delta. \tag{2.1}$$

The blow-up D_X^ν of D_X at the ideal $(w_{11}, w_{12}, w_{13}, \dots, w_{1\delta})$ is smooth.

Proof. Keep Notation 2.1. Let ξ be a spin curve of C supported on the blow-up X of C at the nodes $p_1, \dots, p_h$. Let $\rho\colon D_C \to D_C$ be given by:

$$\rho(t_1, \dots, t_h, t_{h+1}, \dots, t_\delta) = (t_1^2, \dots, t_h^2, t_{h+1}, \dots, t_\delta).$$

We have that $\mathrm{Aut}(\xi)$ acts on D_C as subgroup of the group of automorphisms of D_C, commuting with ρ, as follows. If $h < \delta$, then Σ_X is a graph with one node and h loops. Thus $\mathrm{Aut}(\xi) = \{id\}$ by [CCC, Lemma 2.3.2, Lemma 3.3.1] and hence $\overline{S_g}$ is smooth at ξ. If $h = \delta$, then Σ_X is a graph with two nodes and h edges. Again by [CCC, Lemma 2.3.2, Lemma 3.3.1], we have:

$$\mathrm{Aut}(\xi) = \{id, (t_1, \dots, t_\delta) \xrightarrow{\beta} (-t_1, \dots, -t_\delta)\}.$$

By definition, $D_X = D_C/\mathrm{Aut}(\xi)$. If we set $w_{ij} = t_it_j$ for $1 \leq i \leq j \leq \delta$, then locally analytically at ξ, the equations of D_X are as in (2.1). Now, $\overline{S_g}$ is given by $D_X \times (\mathrm{Def}(C) \cap \mathbb{C}^{3g-\delta-3}_{t_{\delta+1},\dots,t_{3g-3}})$ and $\overline{S_g}$ is singular at ξ.

Let D_X^ν be the blow-up of D_X at the ideal $(w_{11}, w_{12}, \dots, w_{1\delta})$. Cover D_X^ν with δ open subsets $U_1, U_2, \dots, U_\delta$, such that the equation of U_s is:

$$\begin{cases} w_{1i} = \alpha_{is}w_{1s} & 1 \leq i \leq \delta \text{ for } i \neq s \\ w_{ii}w_{jj} = w_{ij}^2 & 1 \leq i < j \leq \delta \\ w_{ii}w_{jj}w_{kk} = w_{ij}w_{jk}w_{ik} & 1 \leq i < j < k \leq \delta \end{cases}$$

for every $s = 1, \dots, \delta$. After few calculations, we get:

$$\begin{cases} w_{is} = \alpha_{is}w_{ss} & 1 \leq i < s \\ w_{si} = \alpha_{is}w_{ss} & s < i \leq \delta \\ w_{ij} = \alpha_{is}\alpha_{js}w_{ss} & 1 \leq i \leq j \leq \delta \text{ for } i, j \neq s \end{cases}$$

In particular, U_s is smooth for every s, hence D_X^ν is smooth. □

Remark 2.3. Keep the notation of Lemma 2.2, with D_X singular. Consider the map $\varphi\colon D_X \to D_C$. Of course, φ is a finite map of degree $2^{\delta-1}$, ramified over the coordinate hyperplanes of D_C. Let $R \subset D_C$ be a line away from the coordinate hyperplanes and containing the origin. By construction, $\varphi^{-1}(R)$ is a union of $2^{\delta-1}$ lines of D_X through the origin, intersecting transversally. The group of the automorphisms of D_X commuting with φ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\delta-1}$ and acts freely and transitively on the set of $2^{\delta-1}$ lines. Let $\nu\colon D_X^\nu \to D_X$ be as in Lemma 2.2 and let $\mathbb{P}_\xi^{\delta-1} = \nu^{-1}(0)$ be the exceptional divisor over the origin. The pull-back to D_X^ν of a line of D_C is a disjoint union of lines, intersecting $\mathbb{P}_\xi^{\delta-1}$. Let $H_{\xi,i} \subset \mathbb{P}_\xi^{\delta-1}$, for $i = 1, \dots, \delta$, be the hyperplane such that the pull-back to D_X^ν of a line contained in $\{t_i = 0\} \subset D_C$ intersects $\mathbb{P}_\xi^{\delta-1}$ in $H_{\xi,i}$. We see that $\mathbb{P}_\xi^{\delta-1} - \cup_{1\leq i\leq\delta}H_{\xi,i}$ is a

$(\mathbb{Z}/2\mathbb{Z})^{\delta-1}\times(\mathbb{C}^*)^{\delta-1}$-torsor. Similarly, for every $\emptyset\neq I\subsetneq\{1,\dots,\delta\}$, we have that $\cap_{i\in I}H_{\xi,i}-\cup_{i\notin I}H_{\xi,i}$ is a $(\mathbb{Z}/2\mathbb{Z})^{\delta-|I|-1}\times(\mathbb{C}^*)^{\delta-|I|-1}$-torsor.

3. Enriched spin curves

In [M], an *enriched stable curve* of a stable curve C with irreducible components $C_1,\dots,C_\gamma$ is defined as $(C,T_{C_1},\dots,T_{C_\gamma})$, where $T_{C_i}=\mathcal{O}_{\mathcal{C}}(C_i)|_C$ and $\mathcal{C}$ is a general smoothing of C. The line bundle T_{C_i} is called a *twister induced by* C_i *and* $\mathcal{C}$. Let $\mathcal{E}_C$ be the set of the enriched stable curves of C. Let D_C be as in Notation 2.1. In the following Lemma, we see that one can obtain a parameter space for $\mathcal{E}_C$, by taking a blow-up of D_C

Proposition 3.1. *Let C be a stable curve with δ nodes and two smooth components C_1 and C_2. Then $\mathcal{E}_C$ forms a $(\mathbb{C}^*)^{\delta-1}$-torsor, which is isomorphic to the $(\mathbb{C}^*)^{\delta-1}$-torsor of linear directions in D_C through the origin, away from the coordinate hyperplanes. The enriched curve corresponding to a line $R\subset D_C$ is (C,T_{C_1},T_{C_2}), where T_{C_1} (resp. T_{C_2}) is the twister induced by C_1 (resp. C_2) and any general smoothing $\mathcal{C}\to B$ of C such that, up to restrict B, the induced map $B\to Def(C)$ has R as image.*

For a proof of Proposition 3.1, see [M, Proposition 3.4, 3.9]. We will need the following result, characterizing the tuples of line bundles which are twisters.

Proposition 3.2. *Let C be a stable curve with irreducible components $C_1,\dots,C_\gamma$ and let $T_1,\dots,T_\gamma$ be line bundles on C. Then $(C,T_1,\dots,T_\gamma)$ is an enriched stable curves of C if and only if the following conditions are satisfied:*

(i) *$T_i\otimes\mathcal{O}_{C_i}\simeq\mathcal{O}_{C_i}(-p_{i,1}-\dots-p_{i,n_i})$ and $T_i\otimes\mathcal{O}_{C_i^c}\simeq\mathcal{O}_{C_i^c}(p_{i,1}+\dots+p_{i,n_i})$ for every $i=1,\dots,\gamma$, where $\{p_{i,1},\dots,p_{i,n_i}\}=C_i\cap C_i^c$.*

(ii) *$\otimes_{i=1}^{\gamma}T_i\simeq\mathcal{O}_C$.*

For a proof of Proposition 3.2, see [M, Proposition 3.16] or [EM, Theorem 6.10]. Similarly, we introduce enriched spin curves, showing that a parameter space for these objects is obtained by the blow-up of D_X described in Lemma 2.2. Recall that, if X is a blow-up of a curve, we denote by $\widetilde{X}=\overline{X-\cup E}$, for E running over the set of exceptional components.

Definition 3.3. Let C be a stable curve with two smooth components. An *enriched spin curve of C supported on X* is given by (X,L_1,L_2), where X is a blow-up of C at a proper subset of nodes and $L_i\in\operatorname{Pic}X$, for $i=1,2$, with $L_i|_E\simeq\mathcal{O}_E(1)$ for every exceptional component E and

$$(L_i|_{\widetilde{X}})^{\otimes 2}\simeq\omega_{\widetilde{X}}\otimes T_{C_i}\,,\ (L_1|_{\widetilde{X}})\otimes(L_2|_{\widetilde{X}})\simeq\omega_{\widetilde{X}}$$

where T_{C_i} is a twister of $\widetilde{X}$ induced by C_i and a general smoothing of $\widetilde{X}$, the same for $i=1,2$. An isomorphism between (X,L_1,L_2) and (X',L_1',L_2') is an isomorphism $\sigma\colon X\to X'$ commuting with the blow-up maps to C and such that $\sigma^*L_i'=L_i$ for $i=1,2$. Denote by $[X,L_1,L_2]$ the isomorphism class of an enriched spin curve, by $\overline{\mathcal{SE}_C}$ the set of the isomorphism classes of enriched spin curves of C, by $\mathcal{SE}_C$ the subset of the ones supported on C.

For every set of indexes I, denote by X_I the blow-up of C at the nodes $\{p_i\}_{i\in I}$ of C. For a smooth curve C, denote by $J_2(C)$ the group of the two-torsion points of the Jacobian variety of C.

Proposition 3.4. *Let C be a curve with δ nodes and two smooth components of genus at least 1. Let C^ν be the normalization of C. Then, for every $I \subsetneq \{1,\ldots,\delta\}$, the set of the isomorphism classes of enriched spin curves of C supported on X_I and $\mathcal{SE}_{\widetilde{X_I}}$ are isomorphic $J_2(C^\nu)\times(\mathbb{Z}/2\mathbb{Z})^{\delta-|I|-1}\times(\mathbb{C}^*)^{\delta-|I|-1}$-torsors.*

Proof. ¿From [C, Lemma 2.1], a class $[X_I, L_1, L_2]$ is determined by the pair $(L_1|_{\widetilde{X_I}}, L_2|_{\widetilde{X_I}})$, hence the set of the isomorphism classes of enriched spin curves of C supported on X_I and $\mathcal{SE}_{\widetilde{X_I}}$ are in bijection. Thus, it suffices to show that $\mathcal{SE}_{\widetilde{X_I}}$ is a $J_2(C^\nu)\times(\mathbb{Z}/2\mathbb{Z})^{\delta-|I|-1}\times(\mathbb{C}^*)^{\delta-|I|-1}$-torsor. The set $\{(\widetilde{X_I}, \omega_{\widetilde{X_I}}\otimes T_{C_1}, \omega_{\widetilde{X_I}}\otimes T_{C_2})\}$ is in bijection with $\mathcal{E}_{\widetilde{X_I}}$, hence by Proposition 3.1 it is a $(\mathbb{C}^*)^{\delta-|I|-1}$-torsor. By definition, $L_1|_{\widetilde{X_I}}$ determines $L_2|_{\widetilde{X_I}}$. For $\omega_{\widetilde{X_I}}\otimes T_{C_1}$ fixed, the set of square roots of $\omega_{\widetilde{X_I}}\otimes T_{C_1}$ is a $J_2(C^\nu)\times(\mathbb{Z}/2\mathbb{Z})^{\delta-|I|-1}$-torsor, because C^ν is the normalization of X_I. □

¿From Proposition 3.4, we get a partition:

$$\overline{\mathcal{SE}_C} = \cup_{I\subsetneq\{1,\ldots,\delta\}}\mathcal{SE}_{\widetilde{X_I}}.$$

Remark 3.5. Let $f\colon \mathcal{X}\to B$ be a smoothing of a nodal curve X and let $\mathcal{N}\in \mathrm{Pic}(\mathcal{X})$. Let $L\in\mathrm{Pic}(X)$ and let ι_0 be an isomorphism $\iota_0\colon L^{\otimes 2}\to\mathcal{N}\otimes\mathcal{O}_X$. By [CCC, Remark 3.0.6], up to shrinking B to a complex neighbourhood of 0, there exists $\mathcal{L}\in\mathrm{Pic}\,\mathcal{X}$ extending L and an isomorphism $\iota\colon\mathcal{L}^{\otimes 2}\to\mathcal{N}$ extending ι_0. Moreover, if $(\mathcal{L}', i')$ is another extension of (L,ι_0), then there is an isomorphism $\chi\colon\mathcal{L}\to\mathcal{L}'$, restricting to the identity, with $\iota = \iota'\circ\chi^{\otimes 2}$.

Keep Notation 2.1 and the notation of Remark 2.3. Let C be a stable curve with two smooth components and δ nodes with $\mathrm{Aut}(C)=\{id\}$. Recall that $D_C = \mathrm{Def}(C)\cap\mathbb{C}^\delta_{t_1,\ldots,t_\delta}$ and $D_X=\varphi^{-1}(D_C)$, where $\varphi\colon\overline{S_g}\to\overline{M_g}$. Let S_C^{sing} be the set of the spin curves of C such that D_X is singular. Recall that S_C^{sing} is described in Lemma 2.2. Now, S_C^{sing} is a $J_2(C^\nu)$-torsor, where C^ν is the normalization of C, then $\cup_{\xi\in S_C^{sing}}(\mathbb{P}^{\delta-1}_\xi - \cup_{1\le i\le\delta}H_{\xi,i})$ is a $J_2(C^\nu)\times(\mathbb{Z}/2\mathbb{Z})^{\delta-1}\times(\mathbb{C}^*)^{\delta-1}$-torsor, where $\mathbb{P}^{\delta-1}_\xi$ and $H_{\xi,i}$ are as in Remark 2.3.

Theorem 3.6. *Let C be a curve with δ nodes and two smooth components of genus at least 1. Assume that $Aut(C)=\{id\}$. Let C^ν be the normalization of C. Then $\mathcal{SE}_C$ and $\cup_{\xi\in S_C^{sing}}(\mathbb{P}^{\delta-1}_\xi - \cup_{1\le i\le\delta}H_{\xi,i})$ are isomorphic $J_2(C^\nu)\times(\mathbb{Z}/2\mathbb{Z})^{\delta-1}\times(\mathbb{C}^*)^{\delta-1}$-torsors.*

Proof. Let C_1, C_2 be the components of C. Pick $(C, L_1, L_2) \in \mathcal{SE}_C$. By the definition of $\mathcal{SE}_C$, there exists a general smoothing $f\colon \mathcal{C} \to B$ of C such that $L_i^{\otimes 2} \simeq \omega_C \otimes T_{C_i}$ for $i = 1, 2$, where T_{C_1} (resp. T_{C_2}) is the twister induced by C_1 (resp. C_2) and $\mathcal{C}$. Let D_C be as in Notation 2.1. Let $R \subset D_C$ be the line through the origin, away from the coordinate hyperplanes, such that, up to restrict B, the induced map $B \to \mathrm{Def}(C)$ has R as image. By Proposition 3.1, the line R does not depend on the chosen smoothing $\mathcal{C} \to B$.

Set $\overline{S}_f(\omega_f) := \overline{S_g} \times_B \overline{M_g}$. Pick the B-curves $\overline{S}_f(\omega_f(C_i))$, for $i = 1, 2$, as in [CCC, Theorem 2.4.1]. Recall that the fiber of $\overline{S}_f(\omega_f(C_i)) \to B$ over $0 \in B$ represents limit square roots of $\omega_f(C_i)|_C$, for $i = 1, 2$. Notice that L_i is a limit square roots of $\omega_f(C_i)|_C$ for $i = 1, 2$. Let $\ell_i \in \overline{S}_f(\omega_f(C_i))$ be the point representing L_i. Since $\omega_f(C_i)$ and ω_f are isomorphic away from the special fiber, the curves $\overline{S}_f(\omega_f(C_i))$ and $\overline{S}_f(\omega_f)$ are isomorphic away from the fiber over $0 \in B$. This implies that they have the same normalization S_f^ν. Call:

$$\psi\colon S_f^\nu \to \overline{S}_f(\omega_f)$$

the normalization. By [CCC, 4.1], $\overline{S}_f(\omega_f(C_i))$ is smooth at ℓ_i for $i = 1, 2$. Therefore S_f^ν and $\overline{S}_f(\omega_f(C_1))$ (resp. S_f^ν and $\overline{S}_f(\omega_f(C_2))$) are isomorphic locally at ℓ_1 (resp. locally at ℓ_2). In particular, we can regard ℓ_1 and ℓ_2 as points of S_f^ν.

We are able to describe $\psi(\ell_1)$ and $\psi(\ell_2)$. Set $C_1 \cap C_2 = \{p_1, \dots, p_\delta\}$. By definition, $L_1 \simeq L_2 \otimes T_{C_1}$. Let $\xi = (X, G) \in S_C^{sing}$ be a spin curve of C, where X is the blow-up of C at the whole set of nodes and G is given by the following data, for every exceptional component E of X:

$$G|_E \simeq \mathcal{O}_E(1)\,,\; G|_{C_i} \simeq (L_i)|_{C_i} \simeq (L_{3-i})|_{C_i}(-\sum_{1 \le s \le \delta} p_s) \text{ for } i = 1, 2. \tag{3.2}$$

Take the Cartesian diagram:

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ & & \downarrow & & \downarrow f \\ & & B' & \xrightarrow{g} & B \end{array}$$

where g is the degree 2 covering of B, totally ramified over 0, and $\mathcal{X}$ is the blow-up at the nodes of C, so that $\mathcal{X}$ is a smoothing of X. Call $\pi\colon \mathcal{X} \to \mathcal{C}$ the composed map. Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be the line bundle of $\mathcal{C}$ such that $\mathcal{L}_1|_C \simeq L_1$ and $\mathcal{L}_1^{\otimes 2} \simeq \omega_f \otimes T_{C_1}$ (resp. $\mathcal{L}_2|_C \simeq L_2$ and $\mathcal{L}_2^{\otimes 2} \simeq \omega_f \otimes T_{C_2}$), as in Remark 3.5. Of course, $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes T_{C_1}$. Set $\mathcal{G}_i := \pi^*\mathcal{L}_i \otimes \mathcal{O}_{\mathcal{X}}(C_{3-i})$, for $i = 1, 2$. By construction, for every exceptional component $E \subset X$ we have:

$$\mathcal{G}_i|_{\widetilde{X}} \simeq G|_{\widetilde{X}}\,,\; \mathcal{G}_i|_E \simeq G|_E \simeq \mathcal{O}_E(1).$$

This implies that L_1 (resp. L_2) is isomorphic to a line bundle G_1 (resp. G_2) in the isomorphism class of ξ. Therefore L_1 and G_1 (resp. L_2 and G_2) are limits of the same family of theta characteristics, hence $\ell_1, \ell_2 \in \psi^{-1}(\xi)$. Since $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes T_{C_1}$, then also L_1 and L_2 are limits of the same family of theta characteristics, hence $\ell_1 = \ell_2 \in \psi^{-1}(\xi) \subset S_f^\nu$. Let $D_X^\nu \xrightarrow{\nu} D_X \xrightarrow{\varphi} D_C$ be as in Remark 2.3. By construction, $\overline{S}_f(\omega_f)$ is given by $\varphi^{-1}(R)$, locally at ξ. In particular, the strict transform $(\nu\circ\varphi)^*(R)$ of R is contained in S_f^ν and $\ell_1 = \ell_2 \in \mathbb{P}_\xi^{\delta-1} - \cup_{1\le i\le\delta} H_{\xi,i}$. Define:

$$\chi\colon \mathcal{SE}_C \longrightarrow \cup_{\xi\in S_C^{sing}}(\mathbb{P}_\xi^{\delta-1} - \cup_{1\le i\le\delta} H_{\xi,i})$$

as $\chi(C, L_1, L_2) := \ell_1 = \ell_2$.

We show that χ is surjective. Consider $\ell \in \mathbb{P}_\xi^{\delta-1} - \cup_{1\le i\le\delta} H_{\xi,i}$, where $\xi = (X, G) \in S_C^{sing}$. Let $R \subset D_C$ be the line corresponding to ℓ. Then $\ell \in (\nu\circ\varphi)^*(R)$, hence $\ell \in S_f^\nu$ and $\psi(l) = \xi$. By [CCC, Lemma 4.1.1], we have $|\psi^{-1}(\xi)| \le 2^{\delta-1}$. Being G fixed, the data (3.2) determine a set $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) of $2^{\delta-1}$ non-isomorphic line bundles represented by $2^{\delta-1}$ different smooth points of $\overline{S}_f(\omega_f(C_1))$ (resp. $\overline{S}_f(\omega_f(C_2))$). Thus $|\psi^{-1}(\xi)| = 2^{\delta-1}$ and the subset of $\overline{S}_f(\omega_f(C_1))$ (resp. $\overline{S}_f(\omega_f(C_2))$) representing $\mathcal{F}_2$ (resp. $\mathcal{F}_2$) is $\psi^{-1}(\xi)$. In particular, ℓ represents two line bundles L_1, L_2 appearing in a enriched spin curve and $\chi(C, L_1, L_2) = \ell$.

We show that χ is injective. Assume that $\chi(C, L_1, L_2) = \chi(C, L_1', L_2')$. In particular, if ℓ_i and ℓ_i' are the points of S_f^ν representing L_i and L_i', for $i = 1, 2$, then $\ell_i = \ell_i'$, which implies that $L_i \simeq L_i'$, for $i = 1, 2$. □

Theorem 3.7. *Let C be a curve with δ nodes and two smooth components of genus at least 1. Assume that $Aut(C) = \{id\}$. Then for every $\emptyset \ne I \subsetneq \{1,\dots,\delta\}$ we have that $\cup_{\xi\in S_C^{sing}}(\cap_{i\in I} H_{\xi,i} - \cup_{i\notin I} H_{\xi,i})$ and $\mathcal{SE}_{\widetilde{X_I}}$ are isomorphic $J_2(C^\nu) \times (\mathbb{Z}/2\mathbb{Z})^{\delta-|I|-1} \times (\mathbb{C}^*)^{\delta-|I|-1}$-torsors.*

Proof. First step. Without loss of generality, let $I = \{1,\dots,h\}$ and X_I be the blow-up of C at the first h nodes. From now on, $\xi = (X, G) \in S_C^{sing}$ will be a fixed spin curve of C, where X is the blow-up of C at the whole set of nodes. Pick $\ell \in \cap_{i\in I} H_{\xi,i} - \cup_{i\notin I} H_{\xi,i}$. Let D_C be as in Notation 2.1. Now, ℓ corresponds to a line of D_C with parametrization:

$$(0, 0, \dots, 0, t_{h+1}, \alpha_{h+2}t_{h+1}, \alpha_{h+3}t_{h+1}, \dots, \alpha_\delta t_{h+1}),$$

for some $\alpha_i \in \mathbb{C}^*$. Consider the curve $R \subset D_C$ with parametrization:

$$(t_{h+1}^2, \dots, t_{h+1}^2, t_{h+1}, \alpha_{h+2}t_{h+1}, \alpha_{h+3}t_{h+1}, \dots, \alpha_\delta t_{h+1}). \tag{3.3}$$

Let $f\colon \mathcal{C} \to B$ be a smoothing of C such that, up to restrict B, the induced map $B \to \mathrm{Def}(C)$ has R as image. Notice that ℓ is contained in the strict transform $(\nu\circ\varphi)^*(R)$ of R. Locally at the first h nodes of C, the surface $\mathcal{C}$ is given by $\{xy - t_{h+1}^2 = 0\} \subset \mathbb{C}^3_{x,y,t_{h+1}}$. Let $\mathcal{X}_I \to \mathcal{C}$ be the resolution of this singularities. The special fiber of $h\colon \mathcal{X}_I \to B$ is X_I and $\mathcal{X}_I$ is smooth. Pick the B-curve $\overline{S}_f(\omega_f) = \overline{S_g} \times_B \overline{M_g}$ and its normalization:

$$\psi\colon S_f^\nu \to \overline{S}_f(\omega_f).$$

Now, $\overline{S}_f(\omega_f)$ is given by $\varphi^{-1}(R)$, locally at ξ, and as in Theorem 3.6, the strict transform $(\nu\circ\varphi)^*(R)$ of R is contained in S_f^ν. In particular $\ell\in\psi^{-1}(\xi)$.

Second Step. Consider the smoothing $h\colon \mathcal{X}_I\to\mathcal{C}$ of X_I. For $i=1,2$, pick the B-curves $\overline{S}_h(\omega_h(C_i))$, as in [CCC, Theorem 2.4.1], which are isomorphic to $\overline{S}_f(\omega_f)$ away from the special fiber. The fiber of $\overline{S}_h(\omega_h(C_i))\to B$ over $0\in B$ represents limit square roots of $(X_I,\omega_h(C_i)|_{X_I})$. In the Second Step, we define points $\ell_i\in\overline{S}_h(\omega_h(C_i))$ such that $\ell_i\in\psi^{-1}(\xi)$, for $i=1,2$.

Pick the following limit square roots of $(X_I,\omega_h(C_i)|_{X_I})$. Let $E_1,\dots,E_h$ be the exceptional components of X_I. For $i=1,2$, let Y_i be the blow-up of X_I at the nodes $C_{3-i}\cap E_1,\dots,C_{3-i}\cap E_h$ and call $F_{3-i,1},\dots,F_{3-i,h}$ the new exceptional components, as in Figure 1. Set $\{p_{h+1},\dots,p_\delta\}:=C_1\cap C_2$.

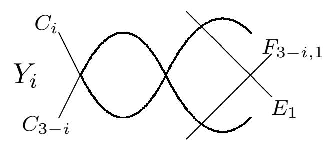

Figure 1

Let (Y_i,L_i), for $i=1,2$, be a limit square root of $(X_I,\omega_h(C_i)|_{X_I})$ defined by the conditions $L_i|_{E_j}\simeq\mathcal{O}_{E_j}$, $L_i|_{F_{3-i,j}}\simeq\mathcal{O}_{F_{3-i,j}}(1)$, for $1\le j\le h$, and:

$$L_i|_{C_i}\simeq G|_{C_i}\ ,\ L_i|_{C_{3-i}}\simeq G|_{C_{3-i}}\Big(\sum_{h<s\le\delta}p_s\Big). \tag{3.4}$$

Let ℓ_i be the point of $\overline{S}_h(\omega_h(C_i))$ representing (Y_i,L_i), $i=1,2$. Since $1\le h<\delta$, the graph Σ_{Y_i} has one node and h loops, $\overline{S}_h(\omega_h(C_i))\to B$ is étale at ℓ_i, $i=1,2$, by [CCC, 4.1]. Thus $\overline{S}_h(\omega_h(C_i))$ and S_f^ν are isomorphic, locally at ℓ_i and we will show that $\ell_i\in\psi^{-1}(\xi)$, $i=1,2$. Take the Cartesian diagram:

$$\begin{array}{ccccccccc}
\mathcal{Z} & \xrightarrow{\pi_1} & \mathcal{Y}_1 & \longrightarrow & \mathcal{X}'_I & \longrightarrow & \mathcal{X}_I & \longrightarrow & \mathcal{C}\\
\downarrow{\scriptstyle\pi_3} & \searrow^{\pi_2} & & \nearrow & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle f} & \\
\mathcal{X} & & \mathcal{Y}_2 & & B' & \xrightarrow{g} & B & &
\end{array}$$

where g is the degree 2 covering of B, totally ramified over 0, $\mathcal{Y}_i\to\mathcal{X}'_I$ is the blow-up at the nodes $C_{3-i}\cap E_1,\dots,C_{3-i}\cap E_h$ for $i=1,2$ and $\mathcal{Z}$ is blow-up at the remaining nodes of X_I. We will specify the map π_3 later. Notice that Y_i is the special fiber of $\mathcal{Y}_i\to B'$ for $i=1,2$ and $\mathcal{Z}$ is smooth. Denote by Z the special fiber of $\mathcal{Z}\to B'$ and let $F_{i1},\dots,F_{ih},E_h\dots,E_\delta$ be the exceptional components of π_i, for $i=1,2$, as in Figure 2.

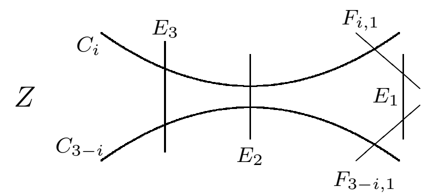

Figure 2

Let $\rho_i\colon Y_i \to X_I$ be the blow-up map and $\mathcal{L}_i \in \mathrm{Pic}(\mathcal{Y}_i)$ be such that:

$$\mathcal{L}_i|_{Y_i} \simeq L_i\,,\ \mathcal{L}_i^{\otimes 2} \simeq \omega_{\mathcal{Y}_i/B'}(-\sum_{1\leq j\leq h} F_{3-i,j}) \otimes \rho_i^*(\mathcal{O}_{\mathcal{X}_I}(C_i)|_{X_I}) \tag{3.5}$$

as in Remark 3.5, for $i=1,2$. The second condition of (3.5) comes from the very definition of limit square root. Let $\pi_3\colon \mathcal{Z} \to \mathcal{X}$ be the contraction of F_{ij} for $i=1,2$ and $j=1,\dots,h$. In particular, the special fiber of $\mathcal{X}$ is the blow-up X of C at the whole set of its nodes. For $i=1,2$, define:

$$\mathcal{G}_i := (\pi_3)_*(\pi_i^*\mathcal{L}_i \otimes \mathcal{O}_{\mathcal{Z}}(C_{3-i} + \sum_{1\leq j\leq h} F_{3-i,j})). \tag{3.6}$$

By construction, $\pi_i^*\mathcal{L}_i\otimes\mathcal{O}_{\mathcal{Z}}(C_{3-i}+\sum_{1\leq j\leq h} F_{3-i,j})$ has degree 0 on each F_{ij}, hence $\mathcal{G}_i$ restricts to a line bundle on X. Furthermore, $\mathcal{G}_i|_{\widetilde{X}} \simeq G|_{\widetilde{X}}$, $\mathcal{G}_i|_E \simeq \mathcal{O}_E(1)$ for every exceptional component $E \subset X$. As in Theorem 3.6, we have that L_i and a line bundle in the equivalence class of $\xi = (X,G)$ are limits of the same family of theta characteristics, hence $\ell_i \in \psi^{-1}(\xi)$, for $i=1,2$.

Third Step. In this Step we define an isomorphism:

$$\chi\colon\ \cup_{\xi\in S_C^{sing}}(\cap_{i\in I}H_{\xi,i} - \cup_{i\notin I}H_{\xi,i}) \longrightarrow \mathcal{SE}_{\widetilde{X_I}}.$$

Now, $|\psi^{-1}(\xi)| \leq 2^{\delta-h-1}$, by [CCC, Lemma 4.4.1], and (3.4) define $2^{\delta-h-1}$ different limit square roots (Y_1,L_1) (resp. (Y_2,L_2)) of $(X_I,\omega_h(C_1))$ (resp. $(X_I,\omega_h(C_2))$). These limit square roots are represented by points of $\psi^{-1}(\xi)$. Hence $|\psi^{-1}(\xi)| = 2^{\delta-h-1}$ and each $l \in \psi^{-1}(\xi)$ represents a limit square root (Y_1,L_1) of $(X_I,\omega_h(C_1))$ and a limit square root (Y_2,L_2) of $(X_I,\omega_h(C_2))$. Define $\chi(\ell) = [\widetilde{X_I}, L_1|_{\widetilde{X_I}}, L_2|_{\widetilde{X_I}}]$. First of all, we show that $\chi(\ell) \in \mathcal{SE}_{\widetilde{X_I}}$. Set $q_{3-i,j} := C_{3-i}\cap F_{3-i,j} \in C_{3-i}$ for $i=1,2$ and $1\leq j\leq h$. The definition of limit square root implies:

$$(L_i|_{\widetilde{X_I}})^{\otimes 2} \simeq \omega_h(C_i)|_{\widetilde{X_I}}(-\sum_{1\leq j\leq h} q_{3-i,j}) \simeq \omega_{\widetilde{X_I}} \otimes \mathcal{O}_{\mathcal{X}_I}(C_i)|_{\widetilde{X_I}}(\sum_{1\leq j\leq h} q_{ij}),$$

for $i=1,2$. Set $M_i = \mathcal{O}_{\mathcal{X}_I}(C_i)|_{\widetilde{X_I}}(\sum_{1\leq j\leq h} q_{ij})$, for $i=1,2$. We have:

$$M_1\otimes M_2 \simeq \mathcal{O}_{\mathcal{X}_I}(C_1+C_2)|_{\widetilde{X_I}}(\sum_{1\leq j\leq h}(q_{ij}+q_{3-i,j})) \simeq \mathcal{O}_{\widetilde{X_I}}$$

$$M_i\otimes\mathcal{O}_{C_j} \simeq \begin{cases}\mathcal{O}_{C_j}(-\sum_{h<s\leq\delta} p_s) & i=j\\ \mathcal{O}_{C_j}(\sum_{h<s\leq\delta} p_s) & i\neq j\end{cases}$$

for $i=1,2$. By Proposition 3.2, M_i is a twister T_{C_i} of $\widetilde{X_I}$ induced by C_i and a general smoothing of $\widetilde{X_I}$, the same for $i=1,2$. To prove the second condition of an enriched spin curve, take the families $\mathcal{Y}_1$ and $\mathcal{Y}_2$, which are the same family away from the special fibers. Let $\theta_i\colon \mathcal{Y} \to \mathcal{Y}_i$ be the blow-up of $\mathcal{Y}_i$ at $C_i\cap E_1,\dots,C_i\cap E_h$, for $i=1,2$, and call Y its special fiber. Since (Y_1,L_1) and (Y_2,L_2) are represented by the same point of S_f^ν, up to change L_2 in the isomorphism class of (Y_2,L_2), we have that L_1 and L_2 are limits of the same family of theta characteristics, given by the line bundles $\mathcal{L}_i$ of (3.5). Set $\mathcal{N} := (\theta_1^*\mathcal{L}_1) \otimes (\theta_2^*\mathcal{L}_2)$ and $\mathcal{N}^* := \mathcal{N}^*|_{\mathcal{Y}-Y}$. Thus, $\mathcal{N}^* \simeq \omega_{\mathcal{Y}-Y/B'-0}$, hence $\mathcal{N} \simeq \omega_{\mathcal{Y}/B'} \otimes \mathcal{O}_{\mathcal{Y}}(D)$, where D is a Cartier divisor supported on components of Y. By (3.4), $\mathcal{N}|_{C_i} \simeq \omega_{\mathcal{Y}/B'} \otimes \mathcal{O}_{C_i}(-\sum_{1\le j\le h} q_{ij})$, thus:

$$(L_1 \otimes L_2)|_{\widetilde{X_I}} \simeq \mathcal{N}|_{\widetilde{X_I}} \simeq \omega_{\mathcal{Y}/B'} \otimes \mathcal{O}_{\widetilde{X_I}}(-\sum_{1\le j\le h}(q_{ij}+q_{3-i,j})) \simeq \omega_{\widetilde{X_I}}.$$

Then, $[\widetilde{X_I}, L_1|_{\widetilde{X_I}}, L_2|_{\widetilde{X_I}}] \in \mathcal{SE}_{\widetilde{X_I}}$.

We conclude by showing that χ is a bijection. The injectivity of χ is trivial. In fact, if we give (Y_i, L_i) and (Y_i, L_i') such that $L_i|_{\widetilde{X_I}} \simeq L_i'|_{\widetilde{X_I}}$, for $i = 1, 2$, then (Y_i, L_i) and (Y_i, L_i') define the same limit square root, for $i = 1, 2$. To show that χ is surjective, we show that the image of χ has the right cardinality. Indeed, an element of the image is determined by choosing ξ in the set S_C^{sing}, which is a $J_2(C^\nu)$-torsor, by choosing $R \subset D_C$ in the set of curves with parametrization as in (3.3), which is a $(\mathbb{C}^*)^{\delta-h-1}$-torsor and l in the set $\psi^{-1}(\xi)$, which is a $(\mathbb{Z}/2\mathbb{Z})^{\delta-h-1}$-torsor. □

Example 3.8. Consider a stable curve C with two smooth components C_1, C_2 and three nodes. Set $C_1 \cap C_2 = \{p_1, p_2, p_3\}$. Assume that $\mathrm{Aut}(C) = \{id\}$. For every spin curve ξ of C, let $\mathbb{P}^2_\xi, H_{\xi,1}, H_{\xi,2}, H_{\xi,3}$ be as in Remark 2.3. Let X_i be the blow-up of C at p_i with exceptional component E_i, for $i = 1, 2, 3$ and let X_{ij} the blow-up at $\{p_i, p_j\}$, with exceptional components E_i, E_j for every $\{i, j\} \subset \{1, 2, 3\}$. Let S_C^{sing} be the set of spin curves of Theorem 3.6 and Theorem 3.7. The set $\overline{\mathcal{SE}_C}$ of enriched spin curves of C is stratified as shown in Figure 3, where ξ runs over the set S_C^{sing}.

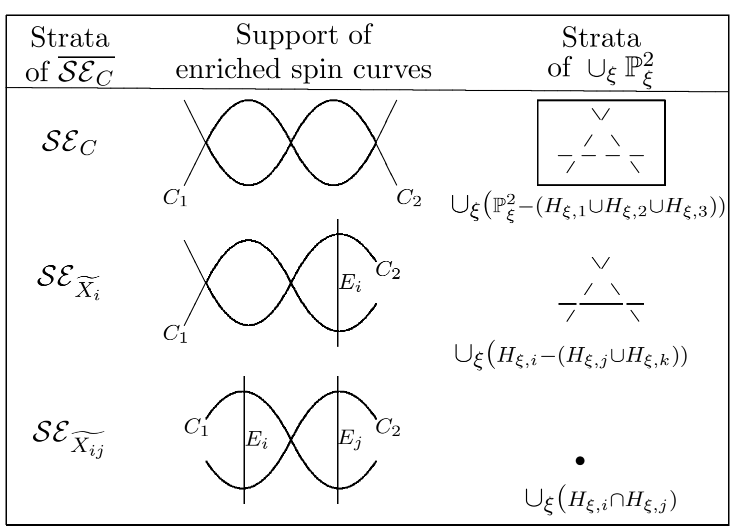

Figure 3

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Marco Pacini
Universidade Federal Fluminense (UFF)
Rua Mário Santos Braga S/N
Niterói-Rio de Janeiro-Brazil
email: pacini@impa.br