Degree-2 Abel maps for nodal curves

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- A **regular local smoothing** of a curve C is a proper and flat morphism $f\colon \mathcal{C} \to B = Spec(k[[t]])$ whose fibers are curves, where f is generically smooth, $f^{-1}(0) \simeq C$ and $\mathcal{C}$ is smooth.

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$$\deg(I \otimes \mathcal{O}_Y)/Tors \geq -\frac{\#(Y \cap \overline{C - Y})}{2}$$

  and it is strict if $P \in Y$.

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- the degree-2 Abel map

$$\alpha_f^2\colon \mathcal{C}^2 - - > J_f^\sigma$$

sending (Q_1, Q_2) on the generic fiber $\mathcal{C}_\eta$ of f to

$$\mathcal{O}_{C_\eta}(2\sigma(\eta) - Q_1 - Q_2)$$

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Theorem (Busonero, Kass, Melo–Viviani)

The B-smooth locus of J_f^σ is the locus of J_f^σ parametrizing invertible sheaves and it is isomorphic to $N(J_\eta)$.

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J_f^σ is a fine moduli space $\Rightarrow$ $\alpha_f^2|_{\dot{\mathcal{C}}^2}$ is induced by an invertible sheaf $\mathcal{L}$ on $\dot{\mathcal{C}}^2 \times_B \mathcal{C}/\dot{\mathcal{C}}^2$, σ-quasistable on the fibers.

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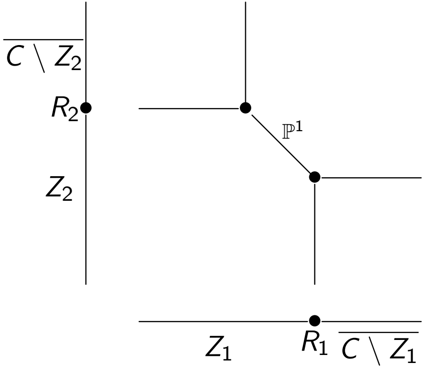

$\phi\colon \widetilde{\mathcal{C}}^2 \to \mathcal{C}^2$

(R_1, R_2) pair of nodes of C

$R_i \in Z_i \cap \overline{C \setminus Z_i}$

$\phi^{-1}(R_1, R_2) \simeq \mathbb{P}^1 \subset \phi^{-1}(Z_1 \times Z_2)$

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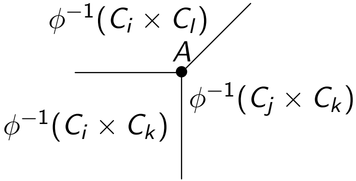

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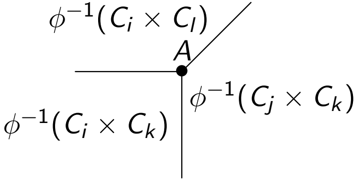

We say that A is *quasistable* if either $R_1 = R_2$ or there are no 2-tails and 3-tails Z such that

$$\{R_1, R_2\} \subseteq Z \cap \overline{C \setminus Z} \text{ and } C_i \cup C_k \subseteq Z$$

The local criterion

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\begin{array}{ccccccccc}
\mathcal{L} & \longrightarrow & \dot{\mathcal{C}}^2 \times_B \mathcal{C} & \longrightarrow & \mathcal{C}^3 & \longleftarrow & \widetilde{\mathcal{C}}^2 \times_B \mathcal{C} & \overset{?}{\longleftarrow} & \mathcal{I} \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \pi & & \\
J_f^{\sigma} & \xleftarrow{\alpha_f^2|_{\dot{\mathcal{C}}^2}} & \dot{\mathcal{C}}^2 & \longrightarrow & \mathcal{C}^2 & \xleftarrow{\phi} & \widetilde{\mathcal{C}}^2 & \overset{?}{\underset{\alpha_{\mathcal{I}}}{\longrightarrow}} & J_f^{\sigma}
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Proposition

Fix $\phi\colon \widetilde{\mathcal{C}}^2 \to \mathcal{C}^2$. *Let* $A \in \widetilde{\mathcal{C}}^2$ *be a quasistable point of* $\widetilde{\mathcal{C}}^2$.

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Proposition

Fix $\phi\colon \widetilde{\mathcal{C}}^2 \to \mathcal{C}^2$. Let $A \in \widetilde{\mathcal{C}}^2$ be a quasistable point of $\widetilde{\mathcal{C}}^2$. Then there is an open subset $U \subset \widetilde{\mathcal{C}}^2$ with $A \in U$ and torsion-free rank-1 sheaf $\mathcal{I}_U$ on $U \times_B \mathcal{C}/U$ which is σ-quasistable on the fibers of π and isomorphic to $\mathcal{L}$ over $\dot{\mathcal{C}}^2 \cap U$.

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Theorem

Let $\phi\colon \widetilde{\mathcal{C}}^2 \to \mathcal{C}^2$ be the blowup of $\mathcal{C}^2$ first along its diagonal subscheme and then along products $Z \times Z$, where Z is a 2-tail or a 3-tail of C.

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J_f^σ is a fine moduli scheme $\Rightarrow$

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J_f^σ is a fine moduli scheme $\Rightarrow$ there is a morphism $\overline{\alpha_f^2}\colon \widetilde{\mathcal{C}}^2 \to J_f^\sigma$ restricting to $\alpha_f^2|_{\dot{\mathcal{C}}^2}$ over $\dot{\mathcal{C}}^2$ (a resolution of the degree-2 Abel map α_f^2).