The general plane quartic is determined by its flex lines

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Theorem (Caporaso-Sernesi, 2003)

Let C be a **general** *smooth plane quartic defined over $\mathbb{C}$. Then C is determined by its bitangents.*

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Let C be a smooth plane curve of degree d defined over $\mathbb{C}$.
(i) If $d = 3$, then C is determined by its 9 flex lines.
(ii) If $d = 4$ and C is general, then C is determined by its 24 flex lines and one flex point.

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- Our strategy follows a degenerative argument.

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Up to scalars, $\alpha\beta\gamma$ is the unique polynomial of degree 3 vanishing on $\mathcal{F}I(F)$.

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$$Aut(\mathcal{F}I(F)) < \{\sigma\in PGL_3(\mathbb{C}) : \sigma \text{ has a non-zero entry}$$

$$\text{in each row and column}\}.$$

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If $C = F^\sigma$, with $\sigma \in PGL_3(\mathbb{C})$, then we get the contradiction

$$\sigma \in Aut(\mathcal{F}l(F)) \setminus Aut(F).$$

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and one can prove that it is not possible.

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- We do not know if any one of (1) and (2) is true !

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Proposition

If $\mathcal{F}I(C)=\mathcal{F}I(V)$, for some smooth plane quartic C, then $C=V$.

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Lemma

We have $\mu_0(\mathcal{F}l(V)) = 4$, $\mu_1(\mathcal{F}l(V)) = 2$ and $\mu_{con}(\mathcal{F}l(V)) < 24$.

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(4) if C is non-reduced and contains a linear component, then either $\mu_0(\Sigma) \geq 5$, or $\mu_1(\Sigma) \geq 3$.

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For a plane line L, define

$$\mathcal{V}_L^{flex} := \{[C] \in \mathbb{P}^{14} : C \text{ is smooth and } L \cdot C = 3p + q\}$$

$$\mathcal{V}_L^{hflex} := \{[C] \in \mathbb{P}^{14} : C \text{ is smooth and } L \cdot C = 4p\}$$

Recovering the general plane quartic

Lemma

If $L \cdot C = 3p + q$, with $p \neq q$, then

$$T_{[C]}\mathcal{V}_L^{flex} \simeq H^0(C, \mathcal{O}_C(4) \otimes \mathcal{O}_C(-2p)).$$

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The morphism

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sending a smooth plane quartic to its configuration of flex lines is generically injective, i.e. the general smooth plane quartic is uniquely determined by its configuration of flex lines.

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