On Abel maps for curves of compact type

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- A curve C is of **compact type** if $C - p$ is not connected, for every node $p \in C$.
- Let $B = \text{Spec}R$, R a DVR and C be a curve. A **smoothing** of C $f : \mathcal{C} \to B$ of C is a proper and flat morphism f, such that $C = f^{-1}(0)$, $f^{-1}(b)$ smooth if $0 \neq b \in B$, and $\mathcal{C}$ smooth.

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- What does it happen if C is singular?

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- Abel maps for a family of curves: limit linear systems on a singular curve (**Eisenbud-Harris, Osserman**).

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- $L \in \mathrm{Pic}^d C$ is **semistable** if for every $\emptyset \neq Y \subsetneq C$:

$$|\deg L|_Y - \frac{d}{2g-2} \deg \omega_C|_Y| \leq \frac{\#(Y \cap \overline{C - Y})}{2}.$$

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- $X \subseteq C$ is **central** (**semicentral**) if $g_Z \leq g/2$ ($g_Z < g/2$) for every connected component Z of $\overline{C - X}$.
- If g is even, let $\Delta_{g/2}$ be the set of the stable curves C of compact type such that $C = C_1 \cup C_2$ and $g_{C_1} = g_{C_2} = g/2$.

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Let $f : \mathcal{C} \to B$ be a smoothing of a stable curve of compact type C. Then there exists an extension $\alpha_f^1 : \mathcal{C} \to J_f^1$ of the first Abel map such that $\alpha_f^1|_C$ factors via $J_C^{1,X^{pr}} \hookrightarrow J_C^1$.

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$$J^{1,X^{pr}}_C = J^{e_1}_C := \{L \in Pic^1 C : \deg_{X^{pr}} L = 1\, , \deg_X L = 0 \text{ for } X \neq X^{pr}\}$$

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- For every $d > 1$, we could define inductively:

$$\mathcal{C}^d := \mathcal{C} \times_B \mathcal{C} \cdots \times_B \mathcal{C} \xrightarrow{(\alpha_f^{d-1}, \alpha_f^1)} J_{\mathcal{C}}^{d-1} \times J_{\mathcal{C}}^1 \xrightarrow{\otimes} J_{\mathcal{C}}^d$$

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- Goal: we want that $\mathrm{Image}(\alpha_f^d|_{C^d}) \subseteq J_C^{d,X^{pr}}$.

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- For an irreducible component X of C and $\underline{d} \in \mathbb{Z}^d$, define:

$$\mathcal{T}_{\underline{d}}(X) := \{Z \subset C : Z \text{ is a } \underline{d}\text{-big tail of } C \text{ and } Z \not\supset X\}.$$

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then $\alpha_f^d|_{C^d} : C^d \to J_C^{\underline{e}_d} \subseteq J_C^{d, X^{pr}}$, where

$$J_C^{\underline{e}_d} := \{L \in Pic^d C : \deg_{X_j} = e_{d,j}, \text{ for } j = 1, \ldots, \gamma\}.$$

Computational aspects and examples

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  For every irreducible non-principal component X of C let $Z(X)$ be the unique subcurve $Z(X) \subseteq \overline{C - X}$ such that $g_{Z(X)} \geq g/2$ (if $g_X = 0$, use that C is stable).

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Otherwise by Third Step, we get the contradiction:

$$\cdots \subsetneq Z(X_j) \cdots \subsetneq Z(X_2) \subsetneq Z(X_1)$$

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$$\psi_d \circ \alpha_f^{d,1}\, ,\, \psi_d \circ \alpha_f^{d,2} : \mathcal{C}^d \longrightarrow P_C^d$$

have the same set-theoretic fibers.

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