Étale models of moduli spaces of theta characteristics

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- We will assume that a curve has genus $g \geq 3$.
- A **semistable** (resp. **stable**) curve is a nodal curve such that every smooth rational component intersects the rest of the curve in at least 2 (resp. 3) points.
- Let $B = \mathrm{Spec}R$, R a DVR. Let C be a stable curve. A **general smoothing** $f : \mathcal{C} \to B$ of C is a proper and flat morphism f, $C = f^{-1}(0)$, $\mathcal{C}^* \to B^* := B - 0$ smooth, and $\mathcal{C}$ smooth.

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(1) describe a distinguished B-model of S_f^*, étale over B, via combinatorial invariants of C.
(2) give a modular interpretation of such a model.

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- One can see that the maximal étale B-model of S_f^* (w.r.t. inclusion) is the **Néron model** $N(S_f^*)$ of S_f^*.

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- A Néron model $N(X_K)$ of X_K is a smooth, separated, finite type B-scheme, $N(X_K)/K = X_K$ and satisfying the *Néron mapping property*.
  NMP: *For every smooth B-scheme Y and morphism $\phi_K\colon Y_K \to X_K$, there exists a unique extension $\phi\colon Y \to N(X_K)$ of ϕ_K.*

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- If $N(X_K)$ exists, it is unique up to a unique isomorphism.

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Theorem

(Chiodo ('08)). $N(S_f^*)$ is finite over B if and only if the dual graph of C has cycles that always share an even number of edges.

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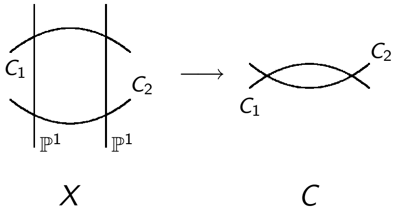

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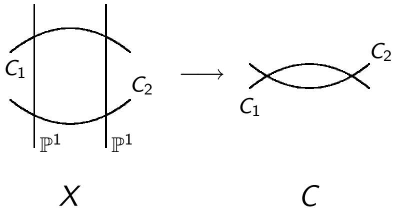

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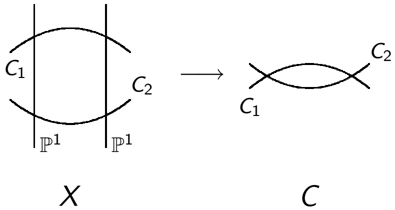

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- A **spin curve** of C is $\xi = (X_\xi, L, \alpha)$, where X_ξ is a blow-up of C, $L \in \mathrm{Pic}(X_\xi)$ and $\alpha\colon L^{\otimes 2} \to \omega_{X_\xi}$ is a homomorphism, such that:
  (1) $L|_E \simeq \mathcal{O}_E(1)$ for each E exceptional.
  (2) α is an isomorphism away from the exceptional components.

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- Let $f : \mathcal{C} \longrightarrow B$ be a general smoothing of a stable curve C and S_f^* be the moduli scheme of theta characteristics on the smooth fibers of f.

- $\overline{S_f}$ compactifies S_f^* over B. $\overline{S_f} \longrightarrow B$ is finite and the fiber over 0 parameterizes the set of (the isomorphism classes of) spin curves of C.

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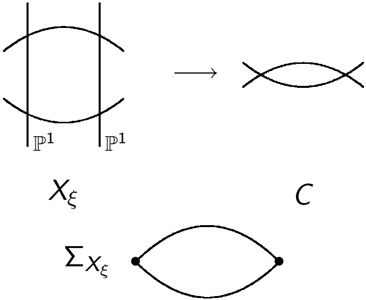

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Theorem

($char(k) = 0$). Let $f : \mathcal{C} \to B$ a general smoothing of a stable curve C with $Aut(C) = \{id\}$. Consider the moduli space $\overline{S_f}$ of spin curves. Let $\nu : \overline{S_f^{\nu}} \to \overline{S_f}$ be its normalization. The following properties are equivalent for any $\xi = (X_\xi, L, \alpha) \in \overline{S_f}$:

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- It is possible to recover the combinatorial result of *Chiodo ('08)*.
- For example, if C has 2 components and 2 nodes, then $N(S_f^*)$ is finite over B.

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- $\mathcal{P}_{d,g}$ is the stack version of the open subset of $\overline{P}_{d,g}$ parameterizing balanced line bundles of degree d on stable curves of genus g.

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- Let $f\colon C \to \{pt\}$ be the trivial family, C a curve. Let p be a smooth point of C. Then I over C is p-quasi-stable w.r.t. $\mathcal{E}$ if for every subcurve $\emptyset \neq Y \subsetneq C$:

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- We have a distinguished subset $\overline{J}^{\sigma}_{\mathcal{E}}$ of $\overline{J}^{d}_{f}$:
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- $\overline{J}^{\sigma}_{\mathcal{E}}$ is a proper B-scheme.

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- Let $p \in C^{sm}$. A f-twister T of C is p-**admissible** if for every $L \in \mathrm{Pic}(C)$ such that $L^{\otimes 2} \simeq \omega_C \otimes T$, then L is p-quasi-stable with respect to $\mathcal{E} = \mathcal{O}_C$.

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$$\min\{a_i\} = 0\,,\ \max\{a_i\} \leq r\ ,\ \}$$

Lemma

Let $f : \mathcal{C} \to B$ be a general smoothing of a stable curve C and $p \in C^{sm}$. Let T be a f-twister of C. Then the following properties are equivalent:
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(ii) there is a unique partition of C into non-empty subcurves $Z_0, \ldots, Z_{r_T}$, $Z_h \cap Z_{h'} \neq \emptyset$ if and only if $|h - h'| \leq 1$ and $p \in Z_0$, such that, if we set $D = \sum_{0 \leq i \leq r_T} i \cdot Z_i$, then $T \simeq \mathcal{O}_{\mathcal{C}}(D)|_C$.

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Theorem

Assume that $Aut(C) = \{id\}$. Let $p \in C^{sm}$.
(1) Then:

$$N(S_f^*) \simeq \frac{\cup_{T \in Ad^1_f(p)} S_f(T)}{\sim},$$

where $\sim$ denotes the gluing along the generic fiber of $S_f(T) \to B$.

Geometric description

Theorem

(2) Let σ be a section of f through the B-smooth locus of $\mathcal{C}$ and set $\mathcal{E} = \mathcal{O}_{\mathcal{C}}$. If $(J_{\mathcal{E}}^{\sigma})^{free}$ is the open subscheme of $J_{\mathcal{E}}^{\sigma}$ parameterizing locally free sheaves, then there exists an immersion:

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Remark

In Busonero ('07) it is shown that $N(Pic^d(\mathcal{C}^*)) \simeq (J^{\sigma}_{\mathcal{E}})^{free}$.

Example

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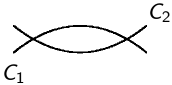
- For example, if C is the stable curve:



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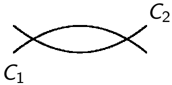

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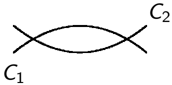

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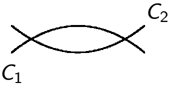

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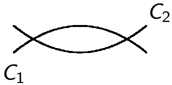

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$$\mathrm{Ad}_f^i(p) = \mathrm{Ad}_f^1(C_1) \text{ if } i \geq 1$$

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- It is easy to check that $N(S_f^*)$ is finite over B.

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