On the n-power map for line bundles over an irreducible curve

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Joint work with E. Esteves

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Escola de Algebra – Rio de Janeiro – August 2008

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$$A_d \colon C^d \to J^d\, C$$

$$(p_1, \ldots, p_d) \to \mathcal{O}_C(p_1 + \cdots + p_d)$$

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- If $n \geq 2$ and p is a node, then $m_p^{\otimes n}$ is not torsion free sheaf. Thus in general u_n does not extend to C.

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Disregarding the minimality of the resolution, the blowup γ_i can be chosen as the blowup along the codimension 2 center Σ_i given by:

$$\Sigma_i := Fitt_1(\Omega^1_{C_i/B}),$$

the subscheme defined by the first Fitting ideal of $\Omega^1_{C_i/B}$.

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- The blow-up X of $C \times_B C$ along Δ is a smooth variety. Let $\tilde{\Delta} \subset X$ be the strict transform of Δ.

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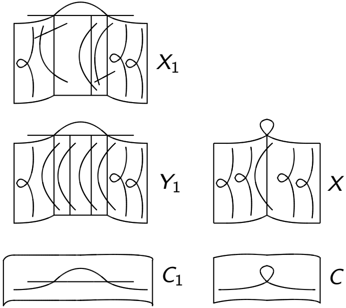

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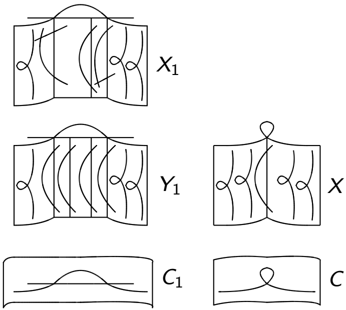

$Y_1 = C_1 \times_C X$ is singular. There are different ways to choose a desingularization. We choose to blow-up Y_1 at $\mathbb{P}^1 \times \mathbb{P}^1$, obtaining the smooth threefold X_1.

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There exists a way to describe a resolution of u_n, combining base changes and blow-ups.

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where: $\psi_n(p_1, \dots p_\delta, L) = (m_{p_1}^{\otimes n}, \dots, m_{p_\delta}^{\otimes n}, L^{\otimes n}).$

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To get a resolution of ψ_n, it suffices to get a resolution of u_n.