Abel maps and powers of line bundles on stable curves

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Introduction

A **curve** is a projective connected, reduced, Gorenstein variety of dimension 1 over $\mathbb{C}$.

C smooth curve.

$$\mathrm{Pic}^d C = \{\text{degree } d \text{ line bundle on } C\} / \text{iso}$$

There is the **Abel map**:

$$A_d \colon C^d \to \mathrm{Pic}^d C$$

defined as:

$$(p_1, \ldots, p_d) \to \mathcal{O}_C(p_1 + \cdots + p_d)$$

Let $p \in C$. If C is not $\mathbb{P}^1$, we have the Abel-Jacobi embedding:

$$A_1 \colon C \to \mathrm{Pic}^1 C \simeq J_C$$

$$q \to \mathcal{O}_C(q) \to \mathcal{O}_C(q - p)$$

Stable curves

Extend the setting to singular curves.

Let C be a **stable curve**, i.e. a nodal curve such that any smooth rational component meets the rest of the curve in at least three points.

Let J_C be the **generalized Jacobian** of C and $p \in C^{sm}$. We can consider:

$$C^{sm} \longrightarrow J_C$$

$$q \to \mathcal{O}_C(q - p)$$

If q is a nodal point, then q is not a Cartier divisor.

We can consider different compactifications of J_C

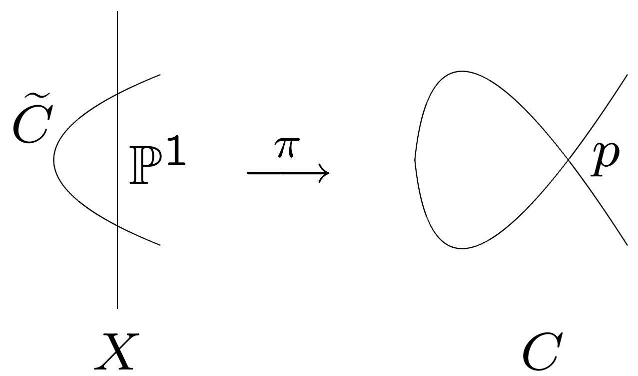
Compactifications of the Picard variety

C irreducible stable curve

Altman-Kleiman's compactification:

$$\overline{J}_C^d = \{\text{rank one torsion free sheaves of degree } d \text{ of } C\}/\text{iso}$$

Blow-up of C and **exceptional component**



Caporaso's compactification: if C irreducible

$\overline{P}_C^d = \{$degree d line bundles on blow-ups of C

with degree 1 on exceptional components $\}$/iso

Relative versions: if $f\colon \mathcal{C} \to B$ is a family of curves, we have Pic_f^d, $\overline{J}_f^d$ and $\overline{P}_f^d$.

Give a completion of the Abel map with values in $\overline{J}_f^d$ or $\overline{P}_f^d$.

Caporaso-Esteves: values in $\overline{P}_f^d$

Caporaso-Coelho-Esteves: values in $\overline{J}_f^d$

The power map

Define the **power map**:

$$\varphi_n : \overline{J}_C^d - - - > \overline{J}_C^{nd}$$

by the rule:

$$I \longrightarrow I^{\otimes n}.$$

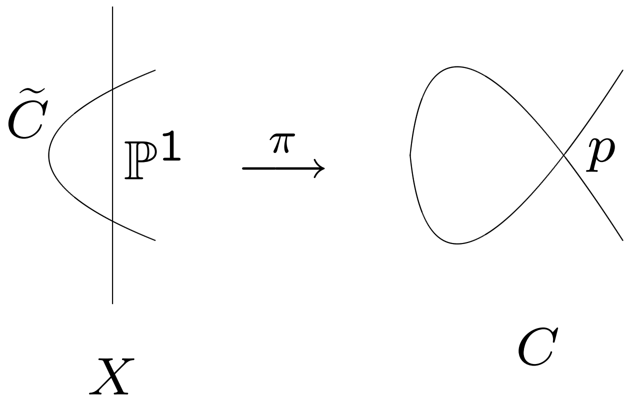
The map is not a morphism: if $I = m_p$, p node, then $I^{\otimes n}$ is not torsion free.

We want to describe a resolution of φ_n:

$$\begin{tikzcd} P_C^d \arrow[d] \arrow[dr, "\widetilde{\varphi}_n"] & \\ \overline{J}_C^d \arrow[r, dotted, "\varphi_n"] & \overline{J}_C^{nd} \end{tikzcd}$$

The idea of the technique used

Let C be a curve with only one node.
Let $X = \widetilde{C} \cup \mathbb{P}^1$ be the **blow-up** in p:



We fix $I \in \overline{J}_C^d$. There exists $L \in \mathrm{Pic}X$ such that:

$$\pi_*(L) = I \quad \deg_{\mathbb{P}^1} L = \begin{cases} 0 & I \text{ invertible} \\ \\ 1 & I \text{ not invertible} \end{cases}$$

If I is not invertible, one can try to define the power by taking $\pi_*(L^{\otimes n})$, but

$$\pi_*(L^{\otimes n}) = I^{\otimes n}$$

has torsion.

Modifications via twisters

To solve the problem we work in families.

Let $f : \mathcal{C} \to B$ be a smoothing of C, i.e. B is a smooth curve and $f^{-1}(0) = C$ for a point $0 \in B$ and C_b smooth for $b \neq 0$.

Take a degree 2 covering $B' \to B$ which is totally ramified over $0 \in B$ and the family $\mathcal{C}' \to B'$, where $\mathcal{C}' = \mathcal{C} \times_B B'$.

Pick the blow-up

$$\pi\colon \mathcal{X} \longrightarrow \mathcal{C}'$$

at the node p of C. Then we get the smoothing $\mathcal{X} \to B'$ of X, the blow-up of C at p.

Assume that $I \in \overline{J}_C^d$ is not invertible.

We take a smoothing of I: let $\mathcal{I}$ a coherent sheaf of $\mathcal{C}'$, flat over B', such that $\mathcal{I}|_C = I$ and $\mathcal{I}|_{C_b}$ is a rank one torsion free sheaf for $b \in B$.

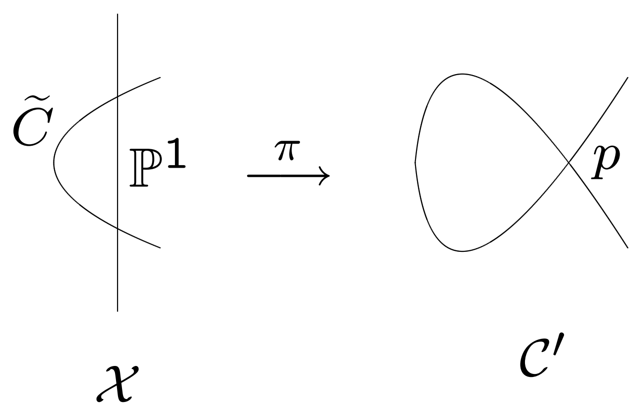
If p is the node of C, then $\pi^{-1}(p) \simeq \mathbb{P}^1$ is a Cartier divisor of $\mathcal{X}$. We can consider the invertible sheaf $\mathcal{M}$ of $\mathcal{X}$ given by:

$$\mathcal{M} = \mathcal{O}_{\mathcal{X}}(\mathbb{P}^1).$$

There exists a $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ such that:

$$\pi_* \mathcal{L} = \mathcal{I}.$$

We have $\deg_{\mathbb{P}^1} \mathcal{L} = 1$.



The construction of a square of I

Up to restrict the base of the family $g : \mathcal{X} \to B'$ we have:

$$X = \mathbb{P}^1 + \widetilde{C} = g^*(0) \sim 0$$

so that:

$$\mathcal{M} = \mathcal{O}_{\mathcal{X}}(\mathbb{P}^1) \simeq \mathcal{O}_{\mathcal{X}}(\mathbb{P}^1 - X) \simeq \mathcal{O}_{\mathcal{X}}(-\widetilde{C}).$$

If $\mathbb{P}^1 \cap \widetilde{C} = \{p_1, p_2\}$, then:

$$\mathcal{M}|_{\widetilde{C}} = \mathcal{O}_{\mathcal{X}}(\mathbb{P}^1)|_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}}(p_1 + p_2)$$

$$\mathcal{M}|_{\mathbb{P}^1} = \mathcal{O}_{\mathcal{X}}(-\widetilde{C})|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-p_1 - p_2)$$

Now:

$$\deg_{\mathbb{P}^1}(\mathcal{L}^{\otimes 2} \otimes \mathcal{M}) = 0.$$

Thus $I^{(2)} := \pi_*((\mathcal{L}^{\otimes 2} \otimes \mathcal{M})|_X)$ is torsion free.

We see $I^{(2)}$ as a square power of I, as it is a limit of a family degenerating also to $I^{\otimes 2}$.

Simplifications

Let C be a nodal curve with only one node.

Let $\mathcal{C} \to B$ be a family having C as a special fiber over $0 \in B$. Consider the power map:

$$\varphi_n \colon \overline{J}^{d}_{\mathcal{C}/B} - -- > \overline{J}^{nd}_{\mathcal{C}/B}$$

There exists a morphism:

$$A \colon \mathcal{C} \times_B J^{d+1}_{\mathcal{C}/B} \to \overline{J}^{d}_{\mathcal{C}/B}$$

given by $A(p, L) = [m_p \otimes L]$, where m_p is the maximal ideal of p.

It is known that A is a smooth morphism.

Using the theory of flat descend, to solve the map φ_n it suffices to solve the map ψ_n in the commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \times_B J^{d+1}_{\mathcal{C}/B} \arrow[r, "A"] \arrow[d, dotted, "\psi_n"'] & \overline{J}^{d}_{\mathcal{C}/B} \arrow[d, dotted, "\varphi_n"] \\
\overline{J}^{-n}_{\mathcal{C}/B} \times_B \overline{J}^{nd+n}_{\mathcal{C}/B} \arrow[r, dotted] & \overline{J}^{nd}_{\mathcal{C}/B}
\end{tikzcd}$$

where:

$$\psi_n(p, L) = (m_p^{\otimes n}, L^{\otimes n}).$$

The second component of ψ_n is a morphism, then it suffices to solve the map:

$$\chi_n : \mathcal{C} - - - > \overline{J}^{-n}_{\mathcal{C}/B}$$

given by: $\chi_n(p) = m_p^{\otimes n}$.

The square power

Let C be an irreducible curve with one node p and $\mathcal{C} \to B$ a smoothing of C with $\mathcal{C}$ smooth.

The threefold $\mathcal{C} \times_B \mathcal{C}$ is singular on (p,p). Let Δ be the diagonal of $\mathcal{C} \times_B \mathcal{C}$.

$$\begin{array}{ccccc} \tilde{\Delta} & \longrightarrow & \mathcal{Y}_1 & & \\ & & \Big\downarrow f & & \\ \Delta & \longrightarrow & \mathcal{C} \times_B \mathcal{C} & & \\ & & \Big\downarrow & & \\ & & \mathcal{C} & \xrightarrow{\chi_1} & \overline{J}^{-1}_{\mathcal{C}/B} \end{array}$$

The blow-up $\mathcal{Y}_1$ of $\mathcal{C} \times_B \mathcal{C}$ along Δ is a smooth variety. Let $\tilde{\Delta} \subset \mathcal{Y}_1$ be the strict transform of Δ. The morphism $\chi_1 \colon \mathcal{C} \to \overline{J}^{-1}_{\mathcal{C}/B}$ is induced by $f_*\left(\mathcal{O}_{\mathcal{Y}_1}(-\tilde{\Delta})\right)$

To get a resolution of $\chi_2\colon \mathcal{C} \to \overline{J}^{-2}_{\mathcal{C}/B}$, we need other divisors.

Pick the blow-up $\mathcal{X} \to \mathcal{C}$ at p and change the base. Let X be the blow-up of C at p.

$$\begin{tikzcd}
 & \tilde{\Delta} \arrow[r] & \mathcal{Y}_2 \arrow[d, "g"'] & \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] & X \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathcal{Y}_1 \times_{\mathcal{C}} \mathcal{X} \arrow[r] \arrow[d, "h"'] & \mathcal{Y}_1 \arrow[d] \\
 & C \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathcal{C} \times_B \mathcal{X} \arrow[r] \arrow[d] & \mathcal{C} \times_B \mathcal{C} \arrow[d] \\
 & \mathbb{P}^1 \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

Now, $\mathcal{Y}_1 \times_{\mathcal{C}} \mathcal{X}$ is singular. There are different ways to choose a desingularization. The blow-up $\mathcal{Y}_2$ of $\mathcal{Y}_1 \times_B \mathcal{X}$ at $\mathbb{P}^1 \times \mathbb{P}^1$ is smooth.

We find in $\mathcal{Y}_2$ the right modification, but the combinatoric is hard.

For example consider:

$$\mathcal{L}_1 = \mathcal{O}_{\mathcal{Y}_2}(-2\tilde{\Delta} + g^*(\mathbb{P}^1 \times \mathbb{P}^1))$$

$$\mathcal{L}_2 = \mathcal{O}_{\mathcal{Y}_2}(-2\tilde{\Delta} + g^*(C^\nu \times \mathbb{P}^1))$$

$h_*(g_*(\mathcal{L}_1))$ is torsion free on the fibers and gives a resolution of χ_2:

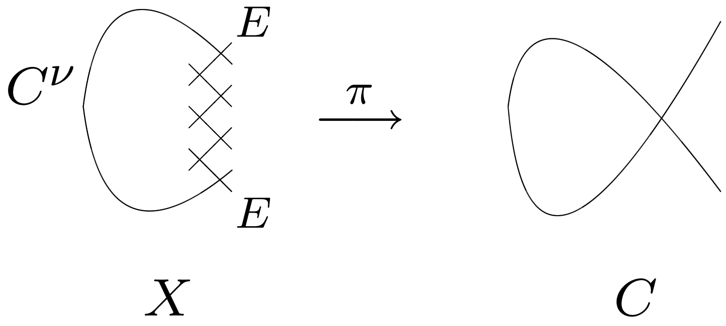
$$\mathcal{X} \longrightarrow \overline{J}^{-2}_{\mathcal{C}/B}$$

$h_*(g_*(\mathcal{L}_2))$ is not torsion free on the fibers.

There exists a way to describe a resolution of χ_n, combining base changes and blow-ups.

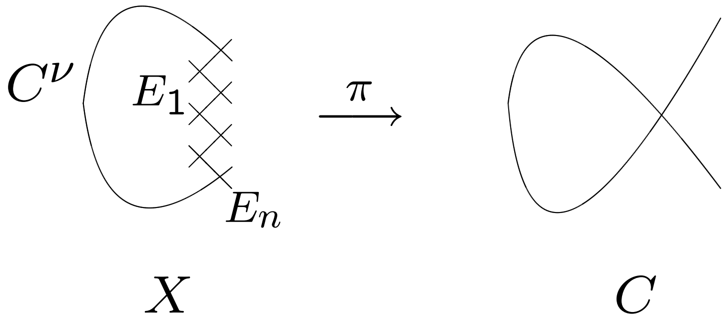
A difference between $\overline{J}_C^d$ and $\overline{P}_C^d$

Let C be an irreducible curve and pick a semistable curve X over C.



A point of $\overline{P}_C^d$ is a line bundle on X whose degrees on the exceptional components is positive and sum-up to 1.

To get a point on $\overline{J}_C^d$, i.e. a torsion free sheaf on C, we have:



Lemma. Let $L \in \mathrm{Pic}X$. Then $\pi_* L$ is a torsion free sheaf on C with $\deg \pi_* L = \deg L$ if and only if:

(a) $\deg_E L \in \{-1, 0, 1\}$, E exceptional;

(b) if $E_1, \ldots, E_n$ chain of exceptional curves, then:

$$\left| \sum_{1 \leq i \leq n} \deg_{E_i} L \right| \leq 1.$$