Q1) (a) $f(x) = \dfrac{x - \sqrt{x^2-1}}{3-x}$

$\lim\limits_{x \to 3^-} \dfrac{x-\sqrt{x^2-1}}{3-x} = \dfrac{3-\sqrt{8}}{0^+} = +\infty$, $3-\sqrt{8} > 0$, pois $\sqrt{8} < 3$

ou

$\lim\limits_{x \to 3^+} \dfrac{x-\sqrt{x^2-1}}{3-x} = \dfrac{3-\sqrt{8}}{0^-} = -\infty \Longrightarrow \boxed{x=3}$ é assíntota vertical

$\lim\limits_{x\to\infty} \dfrac{x-\sqrt{x^2-1}}{3-x} = \lim\limits_{x\to\infty} \dfrac{x-\sqrt{x^2(1-\frac{1}{x^2})}}{3-x} = \lim\limits_{x\to\infty} \dfrac{x - |x|\sqrt{1-\frac{1}{x^2}}}{3-x}$ $\left\{ \begin{array}{l} x>0 \\ |x| = x \end{array}\right.$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{x}\left(1-\sqrt{1-\frac{1}{x^2}}\right)}{\cancel{x}\left(\frac{3}{x}-1\right)} = \dfrac{1-1}{-1} = 0 \Longrightarrow \boxed{y=0}$ é assíntota horizontal

$\lim\limits_{x\to-\infty} \dfrac{x-\sqrt{x^2-1}}{3-x} = \lim\limits_{x\to-\infty} \dfrac{x-|x|\sqrt{1-\frac{1}{x^2}}}{3-x} =$ $\left(\begin{array}{l} x<0 \\ |x| = -x \end{array}\right)$

$= \lim\limits_{x\to-\infty} \dfrac{x + x\sqrt{1-\frac{1}{x^2}}}{3-x} = \lim\limits_{x\to-\infty} \dfrac{\cancel{x}\left(1+\sqrt{1-\frac{1}{x^2}}\right)}{\cancel{x}\left(\frac{3}{x}-1\right)} = \dfrac{1+1}{-1} = -2$

$\Longrightarrow \boxed{y=-2}$ é assíntota horizontal

(b) $\lim\limits_{x\to 0} \underbrace{\dfrac{2x^2-(1-\cos(2x))}{x^2}}_{=f(x)} \cdot \operatorname{sen}\left(\dfrac{1}{x}\right)$

$\lim\limits_{x\to0} f(x) = \lim\limits_{x\to0} 2 - \dfrac{1-\cos(2x)}{x^2} =$

$= \lim\limits_{x\to0} 2 - \dfrac{(1-\cos(2x))}{x^2}\dfrac{(1+\cos(2x))}{1+\cos(2x)} =$

$= \lim\limits_{x\to0} 2 - \dfrac{1-\cos^2(2x)}{x^2} \times \dfrac{1}{1+\cos(2x)} =$

$= \lim\limits_{x\to0} 2 - 4.\dfrac{\operatorname{sen}^2(2x)}{4.x^2} \times \dfrac{1}{1+\cos(2x)} = 2 - 4 \times \dfrac{1}{2} = 2-2 = 0$

obs: podia usar $\lim\limits_{x\to0} \dfrac{1-\cos 2x}{x^2} = \lim\limits_{x\to0} \dfrac{4(1-\cos 2x)}{4x^2} = 4 \cdot \dfrac{1}{2} = 2$

Logo,

$\lim\limits_{x\to0} \overset{\to 0}{f(x)} \underbrace{\operatorname{sen}\left(\dfrac{1}{x}\right)}_{\text{limitada}} = 0$ (pelo Teorema do anulamento)

Q2) $f(x) = \begin{cases} a\sqrt{x+4} & se \quad x \leq 0 \\ 1 + sen(bx) & se \quad x > 0 \end{cases}$

contínua em $x = 0$ se $f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$f(0) = a\sqrt{4} = 2a$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} a\sqrt{x+4} = a.\sqrt{4} = 2a$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 + sen\, bx = 1 + 0 = 1$

Logo, $2a = 1 \Rightarrow \boxed{a = \frac{1}{2}}$

dif em $x = 0$ se $f_+'(0) = f_-'(0)$

sabemos que: f dif em $x=0 \Rightarrow$ f contínua em $x = 0$

Se f não é contínua em $x=0 \Rightarrow$ f não é dif em $x = 0$.

Assim, para que f seja dif em $x = 0$ é preciso que f seja contínua em $x = 0$,

portanto, $a = \frac{1}{2}$ e $f(x) = \begin{cases} \frac{1}{2}\sqrt{x+4} & se \ x \leq 0 \\ 1 + sen(bx) & se \ x > 0 \end{cases}$

e $f_-'(0) = \lim_{x \to 0^-} \frac{f(x) - \overset{=1}{f(0)}}{x - 0} = \lim_{x \to 0^-} \frac{\frac{1}{2}\sqrt{x+4} - 1}{x} =$

$= \lim_{x \to 0^-} \frac{\sqrt{x+4} - 2}{2x} = \lim_{x \to 0^-} \frac{(\sqrt{x+4} - 2)}{2x} \cdot \frac{(\sqrt{x+4} + 2)}{\sqrt{x-4} + 2}$

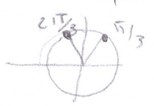
$= \lim_{x \to 0^-} \frac{x + \cancel{4} - \cancel{4}}{2x(\sqrt{x+4}+2)} = \lim_{x \to 0^-} \frac{1}{2(\sqrt{x+4}+2)} = \frac{1}{2(\sqrt{4}+2)} = \frac{1}{8}$

$f_+'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 + sen(bx) - 1}{x} =$

$= \lim_{x \to 0^+} \frac{sen(bx)}{x} = \lim_{x \to 0^+} b\, \frac{sen(bx)}{bx} = b.1 = b$

Logo $\boxed{b = \frac{1}{8}}$

(Q3) (a) $f(x) = \dfrac{\csc x}{2\sqrt{3} + \tan x}$, $x_0 = \dfrac{2\pi}{3}$



Eq. da reta tangente: $y - f\left(\frac{2\pi}{3}\right) = f'\left(\frac{2\pi}{3}\right)\left(x - \frac{2\pi}{3}\right)$

$f(\pi/3) = \dfrac{\csc(2\pi/3)}{2\sqrt{3} + \tan\left(\frac{2\pi}{3}\right)}$

$f\left(\frac{\pi}{3}\right) = \dfrac{\frac{2}{\sqrt{3}}}{2\sqrt{3} - \sqrt{3}} = \dfrac{2/\sqrt{3}}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{1}{\sqrt{3}} = \dfrac{2}{3}$

$\csc\left(\frac{2\pi}{3}\right) = \dfrac{1}{\sin\left(\frac{2\pi}{3}\right)} = \dfrac{1}{\sin\frac{\pi}{3}} = \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}}$

$\tan\left(\frac{2\pi}{3}\right) = \dfrac{\sin 2\pi/3}{\cos\frac{2\pi}{3}} = \dfrac{\sin\pi/3}{-\cos(\pi/3)} = \dfrac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$

$f'(x) = \dfrac{(2\sqrt{3} + \tan x)(-(\csc x)(\cot x)) - \csc x\,(0 + \sec^2 x)}{(2\sqrt{3} + \tan x)^2}$

$f'\left(\frac{2\pi}{3}\right) = \dfrac{(2\sqrt{3} - \sqrt{3})\left(-\frac{2}{\sqrt{3}}\right)\cdot\left(\frac{-1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\,(4)}{(2\sqrt{3} - \sqrt{3})^2}$

$\cot\left(\frac{2\pi}{3}\right) = -\dfrac{1}{\sqrt{3}}$

$\sec\left(\frac{2\pi}{3}\right) = -2$

$= \dfrac{\sqrt{3}\left(\times\frac{2}{3}\right) - \frac{8}{\sqrt{3}}}{3} = \dfrac{\frac{2\sqrt{3}}{3} - \frac{8\sqrt{3}}{3}}{3}$

$= \dfrac{\frac{-6\sqrt{3}}{3}}{3} = \dfrac{-6\sqrt{3}}{9} = \dfrac{-2\sqrt{3}}{3}$

Logo, eq. da reta tangente: $\boxed{y - \frac{2}{3} = \frac{-2\sqrt{3}}{3}\left(x - \frac{2\pi}{3}\right)}$

(b) $r(x) = x \,.\, F(2\,G(\sqrt[3]{x}))$

$r'(x) = x\left[F'(2\,G(\sqrt[3]{x})) \,.\, 2\,G'(\sqrt[3]{x}) \,.\, \dfrac{1}{3\sqrt[3]{x^2}}\right] + 1 \,.\, F(2\,G(\sqrt[3]{x}))$.

$r'(8) = ?$

$(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$

$r'(8) = 8 \,.\, \left[F'(2\,\underbrace{G(2)}_{=3}) \,.\, 2\,\underbrace{G'(2)}_{=-1} \,.\, \dfrac{1}{3\,.\,4^{2}}\right] + F(2\,\underbrace{G(2)}_{=3})$

$r'(8) = 8 \,.\, \underbrace{F'(6)}_{=-2} \,.\, 2 \,.\, G'(2) \,.\, \dfrac{1}{3\,.\,4} + F(6)$

$r'(8) = 8 \,.\, (-2) \,.\, 2 \,.\, (-1) \,.\, \dfrac{1}{3\,.\,4} + 3$

$r'(8) = 8 \,.\, 4 \,.\, \dfrac{1}{3\,.\,4} + 3 = \dfrac{8}{3} + 3 = \dfrac{17}{3}$ //

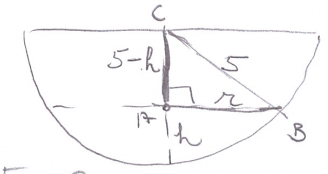
(Q4) Área do espelho d'água $= \pi r^2 = A(r)$.

$\dfrac{dA}{dt}$ = taxa de variação do espelho d'água

$\dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt}$

$\dfrac{dr}{dt}$ = taxa de variação do raio

$\dfrac{dh}{dt}$ = taxa de variação da altura $= 0{,}5\ cm/min$



O triângulo ABC é retângulo, ∴ por Pitágoras,

$$(5-h)^2 + r^2 = 25 \text{ , derivando}$$

$$\cancel{2}(5-h)\cdot\left(-\frac{dh}{dt}\right) + \cancel{2}r\cdot\frac{dr}{dt} = 0$$

$h = 2\ cm$, $3^2 + r^2 = 25 \Rightarrow r^2 = 16$, $\boxed{r = 4\ cm}$

$$(5-2)(-0{,}5) + 4\cdot\frac{dr}{dt} = 0$$

$$4\frac{dr}{dt} = 3 \times 0.5$$

$$\frac{dr}{dt} = 3 \times \frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8}$$

Logo,

$$\frac{dA}{dt} = 2\pi \cdot 4 \times \frac{3}{8} = 3\pi\ cm^2/min$$

//