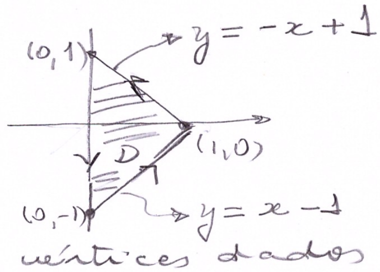
Q1) $\displaystyle\int_C \underbrace{(xy - e^{x^2})}_{P(x,y)}\,dx + \underbrace{(x^2 - y)}_{Q(x,y)}\,dy$



$\vec{F} = (P, Q)$ é de classe C^1.

C é de classe C^1 por partes

D é a região triangular de vértices dados,

$C = \partial D$, orientada positivamente em relação a D.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x = x$$

Pelo Teorema de Green,

$$\int_{C=\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D x\,dx\,dy$$

Cálculo da integral dupla:

$$\iint_D x\,dx\,dy = \int_0^1 \int_{x-1}^{-x+1} x\,dy\,dx = \int_0^1 xy\Big]_{x-1}^{1-x} dx =$$

$$= \int_0^1 x\,((1-x) - (x-1))\,dx = \int_0^1 x(2-2x)\,dx =$$

$$= 2\int_0^1 x - x^2 = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\times\frac{1}{6} = \frac{1}{3}$$

~~Anexo resolvido por integral de linha~~

Q2) $\displaystyle\int_C \vec{F}\cdot d\vec{r} = ?$ $\quad \vec{F}(x,y,z) = (2y - 3z,\ 2x+2z,\ 2y - 3x)$ $\quad \begin{cases} f(0) = 1 \\ f(2\pi) = 2 \end{cases}$

$$\vec{r}(t) = \left(f(t)\cos t,\ f(t)\sin t,\ \frac{t}{\pi}\right)$$

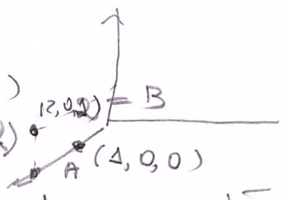
Vamos verificar se $\operatorname{rot}\vec{F} = \vec{0}$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y-3z & 2x+2z & 2y-3x \end{vmatrix} = (2-2,\ -3+3,\ 0-2) = \vec{0}$$

Como $\vec{F}$ é de classe C^1;

$\vec{r}(0) = (\underbrace{f(0)}_{=1}\cos 0,\ \underbrace{f(0)}_{=1}\sin 0,\ \frac{0}{\pi}) = (1,0,0)$

$\vec{r}(2\pi) = (\underbrace{f(2\pi)}_{=2}\cdot\cos 2\pi,\ \underbrace{f(2\pi)}_{=2}\sin 2\pi,\ \frac{2\pi}{\pi}) = (2,0,2)$



Como $\operatorname{rot}\vec{F} = \vec{0}$, pelo Teorema das quatro equivalências, $\int_C \vec{F}\cdot d\vec{r}$, a integral independe da curva que liga A até B, e

$\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A)$ $\quad , \nabla f = F$ (continua...)

Determinando a função potencial f

$$\frac{\partial f}{\partial x}(x,y,z) = 2y - 3z \Rightarrow f(x,y,z) = 2xy - 3xz + \underbrace{c_1(y,z)}_{=2yz}$$

$$\frac{\partial f}{\partial y}(x,y,z) = 2x + 2z \Rightarrow f(x,y,z) = 2xy + 2yz + \underbrace{c_2(x,z)}_{=-3xz}$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2y - 3x \Rightarrow f(x,y,z) = 2yz - 3xz + \underbrace{c_3(x,y)}_{=2xy}$$

$$\Rightarrow f(x,y,z) = 2xy - 3xz + 2yz$$

Logo $\int_C \vec{F}\cdot d\vec{r} = f(2,0,2) - f(1,0,0)$

$$= (0 + 12 + 0) - (0 + 0 + 0) = -12$$

Outra maneira, parametrizando outro caminho entre os pontos $(1,0,0)$ e $(2,0,2)$, por exemplo, o segmento de reta que liga os pontos.

$$\vec{r}(t) = (1,0,0) + t(2-1, 0-0, 2-0), \quad 0 \le t \le 1$$

$$\vec{r}(t) = (1+t, 0, 2t)$$

$$\vec{r}\,'(t) = (1, 0, 2)$$

$$\vec{F} = (2y - 3z, 2x + 2z, 2y - 3x)$$

$$\vec{F}(1+t, 0, 2t) = (-6t, 2 + 2t + 4t, -3 - 3t)$$

$$= (-6t, 2 + 6t, -3 - 3t)$$

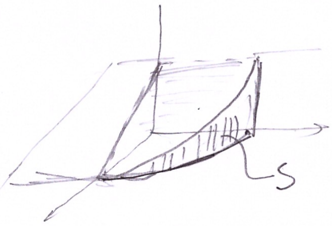
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt =$$

$$= \int_0^1 (-6t - 6 - 6t)\,dt = -\int_0^1 (12t + 6)\,dt$$

$$= -6\int_0^1 (2t+1)\,dt = -6\left(2\frac{t^2}{2} + t\right)\Big|_0^1$$

$$= -6 \times 2 = -12$$

Q3) $x^2+y^2=1$, $x+z=1$



$\varphi(t,z)=(\cos t, \operatorname{sen} t, z)$

$D: \begin{cases} 0 \leq t \leq \pi/2 \\ 0 \leq z \leq 1-\cos t \end{cases}$

Área de $S = \iint_S dS$

$= \iint_D \left\| \frac{\partial\varphi}{\partial t} \times \frac{\partial\varphi}{\partial z} \right\| dz\,dt$

$= \iint_D \sqrt{\cos^2 t + \operatorname{sen}^2 t + 0}\; dz\,dt$

$= \iint_D dz\,dt = \int_0^{\pi/2}\int_0^{1-\cos t} dz\,dt =$

$= \int_0^{\pi/2} (1-\cos t)\,dt = t - \operatorname{sen} t \Big|_0^{\pi/2} = \left(\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{2}\right) - (0-0)$

$= \frac{\pi}{2} - 1$

$x+z=1$

$\cos t + z = 1$

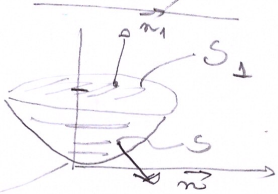
$z = 1-\cos t$

$\frac{\partial\varphi}{\partial t} \times \frac{\partial\varphi}{\partial z} = \begin{vmatrix} \vec i & \vec j & \vec k \\ -\operatorname{sen} t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= (\cos t, \operatorname{sen} t, 0)$

Q4) $x^2+y^2+z^2 = 2az$

$x^2+y^2+(z-a)^2 = a^2$, $z \leq a$



$\vec F = y^5\vec i + x^5\vec j + 2z\vec k$

$\operatorname{div}\vec F = 0+0+2 = 2$

$\vec F$ de classe C^1, $S \cup S_1 = \partial W$, S_1 – plano $z=a$.

$\vec n$ e $\vec n_1$ exterior a W = sólido entre a semi-esfera e o plano.

$\partial W = S \cup S_1$ é regular por partes.

Fluxo de $\vec F$ através de S é $\iint_S \vec F\cdot\vec n\, dS$.

Pelo Teorema de Gauss,

$\iint_{S\cup S_1=\partial W} \vec F\cdot\vec n\, dS = \iiint_W \operatorname{div}\vec F\, dV = 2\underbrace{\iiint_W dV}_{\text{volume de } W}$

$\iint_S \vec F\cdot\vec n\, dS + \iint_{S_1} \vec F\cdot\vec n\, dS = 2$ volume de W

S_1 : $\varphi(x,y,z) = (x,y,a)$

$\frac{\partial\varphi}{\partial x} \times \frac{\partial\varphi}{\partial y} = (0,0,1) \rightarrow$ aponta na direção de $\vec n_1$

$\iint \vec F\cdot\vec n\, dS = \iint (\vec F(x,y,a)\cdot(0,0,1))\, dx\,dy = \iint (y^5, x^5, 2a)\cdot(0,0,1)\, dx\,dy$

continuação da Q4)

$$= \iint_D 2a\,dx\,dy = 2a \iint_D dx\,dy = 2a(\text{área de } D)$$

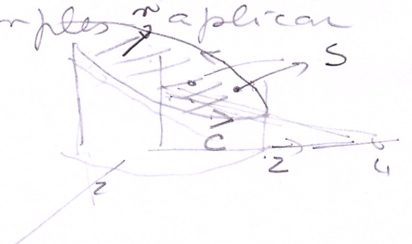
$$D:\ x^2+y^2=a^2$$

Logo $$\iint_S \vec{F}\cdot\vec{n}\,dS + \underbrace{2a(\text{área de } D)}_{=2a\cdot\pi a^2} = \underbrace{2\,\text{volume de } W}_{2\times\frac{1}{2}\times\frac{4}{3}\pi a^3}$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \frac{4}{3}\pi a^3 - 2\pi a^3 = \boxed{-\frac{2}{3}\pi a^3}$$

Q5) $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F}(x,y,z) = (2yz - x^2,\ 2xz - y^2,\ \cos(y^2+z^2))$

Vamos verificar se é simples aplicar o Teorema de Stokes.



$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - x^2 & 2xz - y^2 & \cos(y^2+z^2) \end{vmatrix}$$

$$= (2y\cos(y^2+z^2) - 2y,\ 2y - 0,\ \underbrace{2z - 2z}_{=0})$$

Parametrização de S: S é o plano limitado por C.

$$\varphi(x,y) = (x, y, 2 - \tfrac{1}{2}y) \qquad y + 2z = 4, \quad z = 2 - \tfrac{1}{2}y$$

$$\frac{\partial\varphi}{\partial x}\times\frac{\partial\varphi}{\partial y} = (0, \tfrac{1}{2}, 1)$$

Logo $$\text{rot}\,\vec{F}\cdot\frac{\partial\varphi}{\partial x}\times\frac{\partial\varphi}{\partial y} = (2y\cos(y^2+z^2) - 2y,\ 2y,\ 0)\cdot(0, \tfrac{1}{2}, 1)$$

$$= 0 + 2y\cdot\tfrac{1}{2} + 0 = y$$

Como $\vec{F}$ é de classe C^1, e $C = \partial S$ é de classe C^1, aplicando o Teorema de Stokes.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D y\,dx\,dy, \quad D:\ x^2+y^2 \le 4$$

$$= \int_0^{2\pi}\int_0^2 r\,\text{sen}\,\theta\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r^3}{3}\,\text{sen}\,\theta\,dr\,d\theta =$$

$$= \frac{r^3}{3}\Big|_0^2 \cdot (-\cos\theta)\Big|_0^{2\pi} = 0 \ //$$

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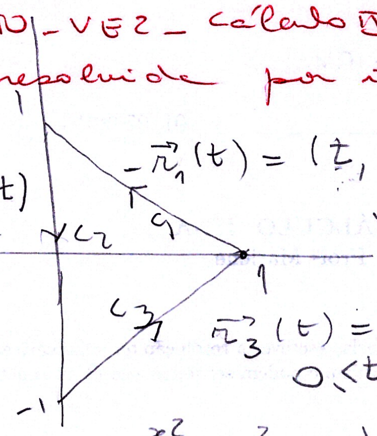
$\boxed{Q1}$ – resolvida por integral de linha (ANEXO)



$-\vec{r}_1(t) = (t, 1-t)$, $\vec{r}_1{}'(t) = (1, -1)$

$0 \leq t \leq 1$

$-\vec{r}_2(t) = (0, t)$

$\vec{r}_2{}'(t) = (0, 1)$

$-1 \leq t \leq 1$

$\vec{r}_3(t) = (t, t-1)$, $\vec{r}_3{}'(t) = (1, 1)$

$0 \leq t \leq 1$

$\vec{F} = (xy - e^{x^2}, x^2 - y)$

$$= -\int_0^1 (t - t^2 - e^{t^2}, t^2 - 1 + t).(1, -1)\, dt \; +$$

$$- \int_{-1}^1 (0 - 1, 0 - t).(0, 1)\, dt \; +$$

$$+ \int_0^1 (t^2 - t - e^{t^2}, t^2 - t + 1).(1, 1)\, dt$$

$$= \int_0^1 (\cancel{-t} + t^2 + \cancel{e^{t^2}} + t^2 \cancel{- 1} \cancel{+ t})\, dt$$

$$+ \underbrace{\int_{-1}^1 t\, dt}_{= 0} + \int_0^1 (t^2 - t \cancel{- e^{t^2}} + t^2 - t \cancel{+ 1})\, dt$$

$$= 4t^2 - 2t \Big|_0^1 = \frac{4t^3}{3} - \frac{2t^2}{2}\Big|_0^1 = \frac{4}{3} - 1 = \frac{1}{3}$$