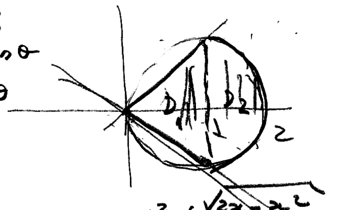
(Q1) (a) $x^2+y^2=2x$ — polares: $r^2 = 2r\cos\theta$, $r = 2\cos\theta$

$x^2+y^2-2x+1 = 1$

$(x-1)^2+y^2=1$

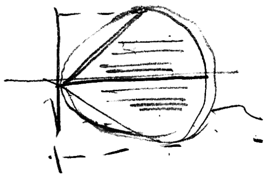
$y^2 = 2x - x^2$

$y = \pm\sqrt{2x-x^2}$



$$\iint_D f(x,y)\,dy\,dx = \int_0^1\int_{-x}^{x} f(x,y)\,dy\,dx + \int_1^2\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x,y)\,dy\,dx$$

$$\iint_D f(x,y)\,dy\,dx = \int_{-1}^0\int_{-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy + \int_0^1\int_{y}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$$



$x = -1-\sqrt{1-y^2}$

$x^2+y^2 = 2x$

$(x-1)^2+y^2 = 1$

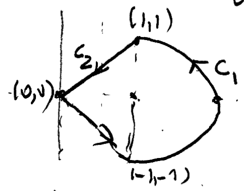
$(x-1)^2 = 1-y^2$

$x-1 = \pm\sqrt{1-y^2}$

$x = +1\pm\sqrt{1-y^2}$

$x \geq 1$, $x = 1+\sqrt{1-y^2}$

(b)



Em polares
$$\int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Fechando com C_3 — segmento de (0,0) para (1,-1)

$C\cup C_3 = \partial D$. $F(x,y) = \left((x^2+y^2)^{3/2}, x\right)$ é classe C^1 em $\mathbb{R}^2 \supset D$.

$C\cup C_3$ é de classe C^1 por partes, pelo Teorema de Green

$$\int_{C\cup C_3=\partial D} \vec F\cdot d\vec r = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - \frac{3}{2}\sqrt{x^2+y^2}\,2y = 1 - 3y\sqrt{x^2+y^2}$$

$$\underbrace{\int_C \vec F\cdot d\vec r}_{?} + \underbrace{\int_{C_3}\vec F\cdot d\vec r}_{\text{calcular}} = \underbrace{\iint_D \left(1-3y\sqrt{x^2+y^2}\right)dx\,dy}_{\text{calcular}}$$

$$\int_{C_3} \vec F\cdot d\vec r = \int_0^1 \left((t^2+t^2)^{3/2}, t\right)\cdot(1,-1)\,dt$$

C_3: $\vec r(t) = (t,-t)$, $0\le t\le 1$, $r'(t)=(1,-1)$

$$= \int_0^1 \left((2t^2)^{3/2} - t\right)dt \qquad 2^{3/2} = \sqrt{2^3} = 2\sqrt2$$

$$= \int_0^1 \left(2\sqrt2\, t^3 - t\right)dt = 2\sqrt2\,\frac{t^4}{4} - \frac{t^2}{2}\Big|_0^1 = \frac{\sqrt2}{2} - \frac12 = \frac{\sqrt2 - 1}{2}$$

continuação da Q1

$$\iint_D \left(1 - 2y\sqrt{x^2+y^2}\right)dxdy = \underbrace{\iint_D dxdy}_{=\text{área de } D} - 2\iint_D y\sqrt{x^2+y^2}\,dxdy$$

função ímpar em y, região simétrica em relação a x, integral $= 0$.

área de $D = \frac{1}{2} \times 2 \times 1 + \frac{1}{2}\pi . 1^2$

$= 1 + \frac{\pi}{2}$

outra maneira de calcular $\iint_D y\sqrt{x^2+y^2}\,dxdy$, em coordenadas polares.

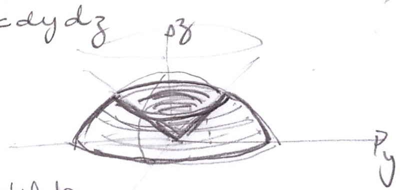
$$\iint_D y\sqrt{x^2+y^2}\,dxdy = \int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} r \operatorname{sen}\theta . r . r\,dr\,d\theta =$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} r^3 \operatorname{sen}\theta\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \frac{r^4}{4}\Big|_0^{2\cos\theta} \operatorname{sen}\theta\,d\theta =$$

$$= \int_{-\pi/4}^{\pi/4} 4\cos^4\theta \operatorname{sen}\theta\,d\theta = -4\,\frac{\cos^5\theta}{5}\Big|_{-\pi/4}^{\pi/4} = \frac{-4}{5}\Big(\cos^5\frac{\pi}{4} - \underbrace{\cos^5\left(\frac{-\pi}{4}\right)}_{=\cos^5\frac{\pi}{4}}\Big) = 0 \;//$$

Logo: $\int_C \vec{F}\,d\vec{r} = 1 + \frac{\pi}{2} - \frac{\sqrt{2}-1}{2} = 1 + \frac{\pi}{2} - \frac{\sqrt{2}}{2} + 1 = 2 + \pi - \frac{\sqrt{2}}{2}$

(Q2) $M = \iiint_W \underbrace{\rho(x,y,z)}_{=kz}\,dxdydz = k\iiint_W z\,dxdydz$



Em coordenadas esféricas.

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 k\,\underbrace{\rho\cos\varphi . \rho^2 \operatorname{sen}\varphi}_{\rho^3 \operatorname{sen}\varphi\cos\varphi}\,d\rho\,d\varphi\,d\theta$$

$x^2+y^2+z^2 = 4$

$z = \sqrt{x^2+y^2}$

$\rho\cos\varphi = \rho \operatorname{sen}\varphi$

$\frac{\operatorname{sen}\varphi}{\cos\varphi} = 1$

$\tan\varphi = 1$

$\varphi = \pi/4$

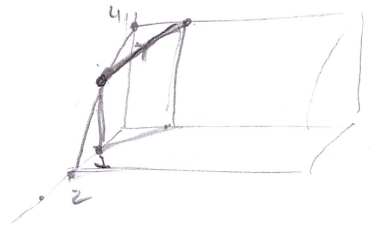
$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} k\,\frac{\rho^4}{4}\Big|_0^2 \operatorname{sen}\varphi\cos\varphi\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} 4k \operatorname{sen}\varphi\cos\varphi\,d\varphi\,d\theta =$$

$$= \int_0^{2\pi} 4k\,\frac{\operatorname{sen}^2\varphi}{2}\Big|_{\pi/4}^{\pi/2} d\theta = \int_0^{2\pi} 2k\Big(1 - \underbrace{\tfrac{1}{2}}_{=1/2}\Big)d\theta$$

$\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$

$= k \times 2\pi \;//$



Q 3) $\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & 2x & 3y \end{vmatrix} =$

$= (3-0, -1-0, 2-0) = (3,-1,2) \neq \vec{0}$

Não pode aplicar o Teo das 4 equivalências.

$C^-: \ \vec{r}(t) = (t, 1-t, 4-t^2)$ $\quad y = 1-x$

$0 \leq t \leq 1$ $\quad z = 4-x^2$

$\vec{F} = (-z, 2x, 3y)$

$\vec{r}\,'(t) = (1,-1,-2t)$

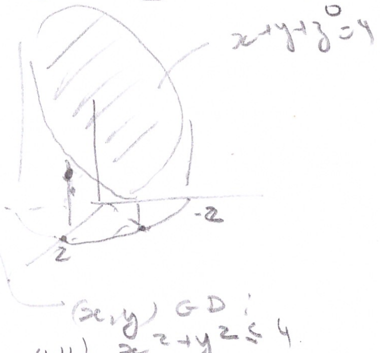
$\int_C \vec{F}\cdot d\vec{r} = -\int_{C^-} \vec{F}\cdot d\vec{r} =$

$= -\int_0^1 (-(4-t^2), 2t, 3-3t)\cdot(1,-1,-2t)\,dt$

$= -\int_0^1 (4+t^2-2t-6t+6t^2)\,dt$

$= -\int_0^1 (7t^2-8t-4)\,dt = -\left(\frac{7t^3}{3} - \frac{8t^2}{2} - 4t\right)\Big]_0^1$

$= -\frac{7}{3} + 4 + 4 = 8 - \frac{7}{3} = \frac{17}{3}$ //



(Q4) $\vec{F}(x,y,z) = (-2x, -2y, 4z)$

(a)

$S: \ \varphi(x,y) = (x, y, 4-x-y)$

$\vec{F}(\varphi(x,y)) = (-2x, -2y, 16-4x-4y)$

$\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = (1,1,1)$

$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D (-2x,-2y,16-4x-4y)\cdot(1,1,1)\,dx\,dy$

$= \iint_D (16-6x-6y)\,dx\,dy$

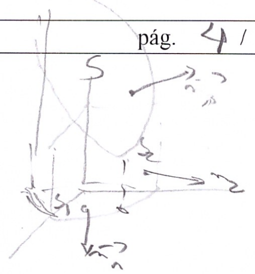
$= \int_0^{2\pi}\int_0^2 (16 - 6r\cos\theta - 6r\,\text{sen}\,\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^2 16r - 6r^2(\cos\theta - \text{sen}\,\theta)\,dr\,d\theta$

$= \int_0^{2\pi} \frac{\cancel{16}^{\,8} r^2}{\cancel{2}} - \frac{\cancel{6}^{\,2} r^3}{\cancel{3}}(\cos\theta - \text{sen}\,\theta)\Big]_0^2 d\theta =$

$\ldots - 16(\text{sen}\,\theta + \cos\theta)\Big]_0^{2\pi}$

(Q4) (b) $\vec{F} = (-2x, -2y, 4z)$



$\text{div}\,\vec{F} = -2-2+4 = 0$

W – sólido delimitado por S_1, S_2 e S

$\vec{F}$ de classe C^1 em $\mathbb{R}^3 \supset W$.

S, S_1, S_2 regulares.

$\partial W = S_1 \cup S_2 \cup S$

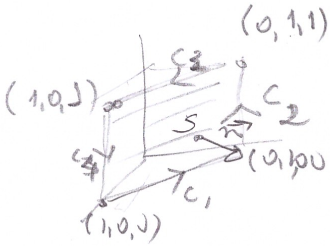
Pelo Teorema de Gauss:

$$\iint_{S \cup S_1 \cup S_2 = \partial W} \vec{F}\cdot\vec{n}\,dS = \iiint_W \underbrace{\text{div}\,\vec{F}}_{=0}\,dV = 0 \Longrightarrow$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{S_1 \cup S_2} \vec{F}\cdot\vec{n}\,dS = 0$$

$$\Rightarrow \iint_{S_1\cup S_2} \vec{F}\cdot\vec{n}\,dS = -\iint_{S_1 \cup S_2} \vec{F}\cdot\vec{n} \overset{(a)}{=} -64\pi \;//$$

(Q5) $C = C_1 \cup C_2 \cup C_3 \cup C_4$



$\vec{F} = (2x, f(y,z), 2y)$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & f(y) & 2y \end{vmatrix} =$$

$= (2-0, 0-0, 0-0) = (2, 0, 0)$

S : $x+y=1$ com fronteira

$\varphi(x,y) = (x, 1-x, z)$, $\begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1 \end{cases}$

$\vec{F}$ de classe C^1 em $\mathbb{R}^3 \supset S$

C de classe C^1 por partes.

Pelo Teorema de Stokes.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{rot}\,\vec{F}\cdot\vec{n}\,dS = -\iint_D (2,0,0)\cdot(-1,-1,0)\,dx\,dy$$

$$\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \underbrace{(-1,-1,0)}_{\text{sentido oposto a } \vec{n}}$

$= 2 \iint_D dx\,dy =$

$= 2$ área de D

$= 2 \times 1 \times 1 = 2 \;//$