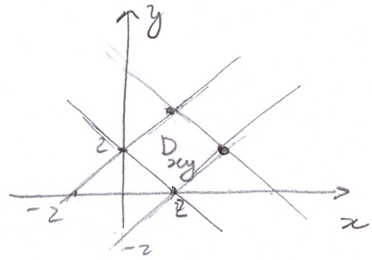
(Q1) $y = x+2 \iff y - x = 2$

$y = x-2 \iff y - x = -2$

$y = 2-x \iff y + x = 2$

$y = 6-x \iff y + x = 6$



$u = y - x \quad , -2 \leq u \leq 2$

$v = y + x \quad \; 2 \leq v \leq 6$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{1}{-1-1} = -\frac{1}{2}$$

$$\iint_{D_{xy}} \cos(y-x)\, dx\, dy = \iint_{D_{uv}} \cos(u) \frac{du\, dv}{2} =$$

$$= \frac{1}{2}\int_2^6 \int_{-2}^{2} \cos(u)\, du\, dv = \frac{1}{2}\int_2^6 + \operatorname{sen} u \Big|_{-2}^{2} dv =$$

$$= \frac{1}{2}\int_2^6 (\operatorname{sen} 2 - \underbrace{\operatorname{sen}(-2)}_{= -\operatorname{sen} 2})\, dv = \frac{2}{2}\int_2^6 (\operatorname{sen} 2)\, dv = (\operatorname{sen} 2)(6-2)$$

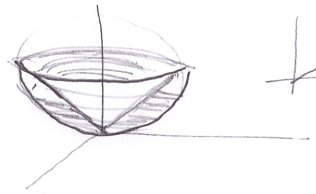
$$= 4 \operatorname{sen} 2 \quad //$$

(Q2) $x^2 + y^2 + (z-1)^2 = 1$

(a) $x^2 + y^2 + z^2 = 2z$

$\rho^2 = 2\rho \cos\varphi$

$\boxed{\rho = 2\cos\varphi}$



$z = \sqrt{x^2+y^2} \quad , \quad z = r$

$\rho\cos\varphi = \rho \operatorname{sen}\varphi \Rightarrow \tan\varphi = 1$

$\varphi = \pi/4$

$$\iiint_W \frac{1}{x^2+y^2}\, dx\, dy\, dz = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\varphi} \frac{1}{\rho^2 \operatorname{sen}^2\varphi}\, \rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \csc\varphi\, d\rho\, d\varphi\, d\theta$$

(b) $\cap$: $z = \sqrt{x^2+y^2} \quad , \quad z = r$

$r^2 + (r-1)^2 = 1$

$r^2 + r^2 - 2r + 1 = 0$

$2r^2 - 2r = 0 \Rightarrow r = 0$

$r = 1$

$x^2+y^2+(z-1)^2 = 1$

$r^2 + (z-1)^2 = 1$

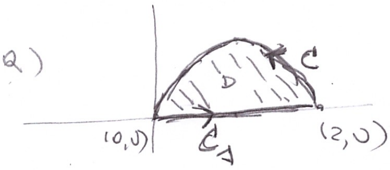
$z = 1 \pm \sqrt{1-r^2}$

$z \leq 1, \boxed{z = 1 - \sqrt{1-r^2}}$

$$\boxed{\int_0^{2\pi}\int_0^{1}\int_{1-\sqrt{1-r^2}}^{r} \frac{1}{r^2}\, r\, dz\, dr\, d\theta}$$

Q3) $x^2+y^2=2x$, $y\geq 0$

$\int_C \underbrace{(x-y^2)}_{P}\,dx + \underbrace{(xy)}_{Q}\,dy$, $\vec{F}=(P,Q)$



$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = y-(-2y) = 3y$

$\vec{F}$ é de classe C^1 em $\underbrace{\mathbb{R}^2}_{\text{aberto}}\supset D$. $\partial D = C_1 \cup C$

Pelo Teorema de Green, $\int_{\partial D = C\cup C_1} P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}_{=3y} dx\,dy$

Logo, $\int_C P\,dx + Q\,dy = \iint_D 3y\,dx\,dy - \int_{C_1} P\,dx + Q\,dy$

D: $x^2+y^2\leq 2x$, $y\geq 0$, passando para coordenadas polares, $r^2 = 2r\cos\theta$

$r = 2\cos\theta$, $0\leq\theta\leq \pi/2$

$\iint_D 3y\,dx\,dy = \int_0^{\pi/2}\int_0^{2\cos\theta} 3\cdot r\,\text{sen}\,\theta\cdot r\,dr\,d\theta = \int_0^{\pi/2} \frac{3r^3}{3}\text{sen}\,\theta\Big|_0^{2\cos\theta} d\theta =$

$= \int_0^{\pi/2} 8\cos^3\theta\,\text{sen}\,\theta\,d\theta = -8\frac{\cos^4\theta}{4}\Big|_0^{\pi/2} = -2[0-1] = 2$ //

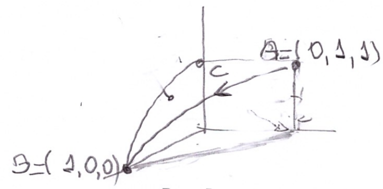
C_1: $\vec{r}(t)=(t,0)$, $\vec{r}\,'(t)=(1,0)$, $0\leq t\leq 2$

$\vec{F}(\vec{r}(t)) = \vec{F}(t,0) = (t-0,0)\,dt$

$\int_C P\,dx + Q\,dy = \int_0^2 (t,0)\cdot(1,0)\,dt = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$ //

Portanto $\int_C P\,dx + Q\,dy = 2-2 = 0$ //

(Q4) $\vec{F}(x,y,z) = (y, x, 2ze^{-z^2})$



$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2ze^{-z^2} \end{vmatrix} =$

$= (0-0, 0-0, 1-1) = \vec{0}$

$\left\{ x^2+z^2=1,\ z=0 \right.$; $x^2=1$, $x>0$, $x=1$, $y=0$

$\left\{ x+y=1,\ x=0 \right.$; $y=1$, $z^2=1$, $z>0$, $z=1$

$\vec{F}$ de classe C^1 em $\mathbb{R}^3$ (simplesmente conexo)

pelo teorema das quatro equivalências,

$\text{rot}\,\vec{F} = \vec{0} \iff \int_A^B \vec{F}\cdot d\vec{r}$ independe do caminho

$\iff \vec{F}$ é conservativo ($\exists$ função potencial $f(x,y)$)

Q4) (continuação)

Uma maneira: determinando $f(x,y)$; $\nabla f = \vec{F}$.

$$f(x,y) = \int y\,dx = xy + C_1(y,z)$$
$$f(x,y) = \int x\,dy = xy + C_2(x,z)$$
$$f(x,y) = \int 2z e^{-z^2} dz = -e^{-z^2} + C_3(x,y)$$
$$\Longrightarrow f(x,y) = xy - e^{-z^2}$$

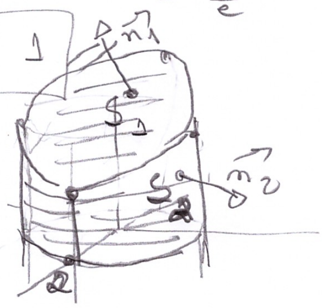
$$\int_{A=(0,1,1)}^{B=(1,0,0)} \vec{F}\cdot d\vec{r} = f(1,0,0) - f(0,1,1) = (0 - \underbrace{e^{-0}}_{=1}) - (0 - \underbrace{e^{-1}}_{=\frac{1}{e}}) = \boxed{\frac{1}{e} - 1}$$

Outra maneira: pela reta C que liga A a B.

$$\vec{r}(t) = A + t(B-A) = (0,1,1) + t[(1,0,0)-(0,1,1)] =$$
$$= (0+t,\ 1-t,\ 1-t) = (t, 1-t, 1-t) \qquad 0 \le t \le 1.$$
$$\vec{r}\,'(t) = (1,-1,-1)$$
$$\vec{F}(\vec{r}(t)) = F(t, 1-t, 1-t) = (1-t,\ t,\ 2(1-t)e^{-(1-t)^2})$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (1-t,\ t,\ 2(1-t)e^{-(1-t)^2})\cdot(1,-1,-1)\,dt =$$
$$= \int_0^1 \left(1 - t - t - 2(1-t)e^{-(1-t)^2}\right) dt = \int_0^1 1 - 2t - 2(1-t)e^{-(1-t)^2} dt$$

Calculando $\int -2(\underbrace{1-t}_{=u})e^{-(1-t)^2} dt = \int -2u\, e^{-u^2}(-1)\,du = -e^{-u^2} + C$

$$\int_C \vec{F}\cdot d\vec{r} = t - \frac{\cancel{2}t^2}{\cancel{2}} - e^{-(1-t)^2}\Big]_0^1 = \Big[(\underbrace{1 - 1 - e^0}_{-1}) - (\underbrace{0 - 0 - e^{-1}}_{= -\frac{1}{e}})\Big]$$
$$= \boxed{\frac{1}{e} - 1}$$



(Q5) $\vec{F}(x,y,z) = (x+2,\ y+f(z),\ 0)$

S_1: $x + z = 4$, $x^2+y^2 \le 4$

S_2: $x^2+y^2 \le 4$, $z \ge 0$,

(a) $\varphi(x,y) = (x,\ y,\ \underbrace{4-x}_{z(x)})$, D: $x^2+y^2 \le 4$

$$\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = (-z_x, -z_y, 1) = (1,0,1) \text{ — mesmo sentido de } \vec{n}_1$$
$$\vec{F}(\varphi(x,y)) = \vec{F}(x, y, 4-x) = (x+2,\ y + f(4-x),\ 0)$$
$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iint_D (x+2,\ y+f(4-x),\ 0)\cdot(1,0,1)\,dx\,dy =$$
$$= \iint_D (x+2)\,dx\,dy = \int_0^{2\pi}\int_0^2 \underbrace{(r\cos\theta + 2)r}_{= r^2\cos\theta + 2r}\,dr\,d\theta =$$
$$= \int_0^{2\pi} \left[\frac{r^3}{3}\cos\theta + \frac{2r^2}{2}\right]_0^2 = \int_0^{2\pi} 4\,d\theta = 4\cdot 2\pi = 8\pi \;//$$

(Q 5) (b) W = sólido delimitado por S_1, S_2, S_3,
onde S_3 é o plano $z=0$, $x^2+y^2\leq 4$.
$\vec{F}$ é de classe C^1 em $\mathbb{R}^3 \supset W$.
Pelo Teorema de Gauss = $\displaystyle\iint_{S=\partial W=S_1\cup S_2\cup S_3} \vec{F}\cdot\vec{n}\, dS = \iiint_W \operatorname{div}\vec{F}\, dV$.

$\operatorname{div}\vec{F} = 1+1+0 = 2$

Logo $\displaystyle\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \iiint_W 2\, dV - \underbrace{\iint_{S_1} \vec{F}\cdot\vec{n}\, dS}_{=8\pi \text{ (item a)}} - \iint_{S_3} \vec{F}\cdot\vec{n}\, dS$

W em coordenadas cilíndricas:
$0\leq\theta\leq 2\pi$, $0\leq r\leq 2$, $0\leq z\leq 4-r\cos\theta$

$$\iiint_W 2\, dV = \int_0^{2\pi}\int_0^2\int_0^{4-r\cos\theta} 2\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \underbrace{2(4-r\cos\theta)\, r}_{=8r-2r^2\cos\theta}\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\cancel{8}^{\,4} r^2}{\cancel{2}} - \frac{2r^3}{3}\cos\theta\right]_0^2 d\theta = \int_0^{2\pi}\left(16 - \frac{16}{3}\cos\theta\right) d\theta =$$

$$= 16\theta - \frac{16}{3}\operatorname{sen}\theta\Big]_0^{2\pi} = 32\pi$$

S_3: $\varphi(x,y) = (x,y,0)$; $\dfrac{\partial\varphi}{\partial x}\times\dfrac{\partial\varphi}{\partial y} = (0,0,1)$ sentido oposto ao de $\vec{n}_3$

$\vec{F}(\varphi(x,y)) = \vec{F}(x,y,0) = (x+2,\, y+f(0),\, 0)$.

$$\iint_{S_3} \vec{F}\cdot\vec{n}\, dS = -\iint_D \underbrace{(x+2,\, y+f(0),\, 0)\cdot(0,0,1)}_{=0}\, dx\,dy = 0$$

Portanto $\displaystyle\iint_{S_2} \vec{F}\cdot\vec{n}\, dS = 32\pi - 8\pi - 0 = 24\pi$ //