On the equation satisfied by a steady Prandtl-Munk vortex sheet

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We show that the vorticity distribution obtained by minimising the induced drag on a wing, the so called Prandtl-Munk vortex sheet, is not a travelling-wave weak solution of the Euler equations, as it is claimed by a number of authors in the literature. However, the Prandtl-Munk vortex sheet is a weak solution of a non-homogeneous equation, where the forcing term represents a "tension" force applied to the tips of the vortex sheet.

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1. Introduction

(a) *Background*

Although vortex sheets cannot exist in Nature, they are important as an ideal model to describe a variety of fluid dynamical phenomena where vortex dynamics plays an important rôle, for instance, in the modelling of roll-up observed in the wakes of bluff bodies or in the trailing vortices of wings.

One of such applications is the determination of the circulation distribution that minimises the induced drag on a plane wing. The calculation can be done in a number of ways, such as using Prandtl's lifting-line theory (cf. Batchelor, 1967) or energy methods (cf. Saffman, 1992). In any case, the result obtained is a 2-D vortex sheet distribution supported on an interval. This distribution is known variously as the Prandtl-Munk vortex sheet or the Elliptically Loaded Wing. This vortex sheet is also related to the so-called Kaffeelöffel thought experiment proposed by Klein (1910), whereby a plate of a certain length is set in uniform motion and then "removed"; a Prandtl-Munk vortex sheet would then be left in the fluid (see Saffman, 1992).

The Prandtl-Munk vortex sheet is the archetypical vortex sheet initial condition that leads to roll-up and, as such, has been extensively studied, although its evolution is not completely understood. In particular, it has been suggested by a number of authors (for instance Krasny, 1987; Meyer, 1982) that the Prandtl-Munk vortex sheet is a travelling-wave weak solution of the Euler equations, which would entail non-uniqueness of weak solutions, given that, in numerical simulations, the vortex sheet is observed to roll-up. It is also known that these 2-D line vortex sheets are prone to a Kelvin-Helmholtz type instability (cf. Moore, 1976), and this instability could be the reason as to why the travelling wave solution is not observed in the numerical experiments (cf. Meyer, 1982). On the other hand, it has also been argued that the singularity in the velocity field at the tips would preclude the Prandtl-Munk vortex sheet from being an equilibrium solution. Indeed, a careful analysis by Saffman (1992) using fluid-dynamical arguments strongly suggests that the Prandtl-Munk vortex could only be in dynamical equilibrium if it had a certain amount of tension applied at its tips. The purpose of the present work is to support Saffman's argument by means of rigorous mathematical analysis. We will show that the Prandtl-Munk vortex sheet gives rise to a travelling-wave weak solution of the inhomogeneous Euler equation with precisely the forcing predicted by Saffman.

We remark that most of the stability analyses of vortex sheets are done through the Birkhoff-Rott equation. Solutions to Birkhoff-Rott are known to be weak solutions of the Euler equations when the curve describing the vortex sheet and its strength are both smooth (see Marchioro & Pulvirenti, 1994). This is not the case for the Prandtl-Munk vortex sheet, which has a singular strength distribution.

(*b*) *Outline*

The remainder of this work is organised as follows: in § 2, we give a precise formulation of the problem, and set up the necessary notation and preliminary results. In § 3 we state the main result and we present its proof. Some conclusions are drawn in § 4.

2. Formulation

In what follows, we will use 2-D Cartesian coordinates given by (x_1, x_2).

Let ω_0 be the initial vorticity describing the Prandtl-Munk vortex sheet:

$$\omega_0(x_1, x_2) = \frac{x_1}{\sqrt{1 - x_1^2}} \chi_{(-1,1)}(x_1) \otimes \delta_0(x_2), \tag{2.1}$$

where $\chi_{(-1,1)}$ represents the characteristic function of the interval $(-1, 1)$ and δ_0 is the Dirac delta at $x_2 = 0$.

The velocity associated to the Prandtl-Munk vortex sheet through the Biot-Savart law, when evaluated by means of a principal value integral at points on the sheet, is constant and equal to $(0, -1/2)$ (cf. Saffman, 1992). Hence we consider the steadily translating vorticity profile given by:

$$\omega(x_1, x_2, t) = \omega_0(x_1, x_2 + \frac{t}{2}). \tag{2.2}$$

The standard weak formulation for the 2-D incompressible Euler equations is given in the following definition:

Definition 1. *A vector field $u \in L^\infty_{loc}([0, \infty); (L^2_{loc}(\mathbb{R}^2))^2)$ is a weak solution of the incompressible 2D Euler equations with initial data $u_0 \in (L^2_{loc})^2$ if:*

1. *The identity div $u = 0$ holds for almost every time in the sense of distributions.*

2. *For any test function $\phi = (\phi_1, \phi_2) \in (C^\infty_c([0, \infty) \times \mathbb{R}^2))^2$ with div $\phi = 0$ we have:*

$$\int_0^\infty \int_{\mathbb{R}^2} (\phi_t \cdot u + u \cdot ((D\phi)u))\, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} \phi(x, 0) \cdot u_0(x)\mathrm{d}x =$$

$$-\int_0^\infty \int_{\mathbb{R}^2} F \cdot \phi \,\mathrm{d}x\mathrm{d}t,$$

where $D\phi$ is the Jacobian matrix of the vector field ϕ.

Although the vorticity associated to the Prandtl-Munk vortex sheet has the particularly simple form (2.1), the associated velocity is fairly complicated. For this reason, we choose to work with the vorticity equation and we will use the weak vorticity formulation, as introduced by Schochet (1995), given below:

Definition 2. *A measure $\omega \in L^\infty([0,\infty); \mathcal{BM}(\mathbb{R}^2) \cap H^{-1}_{loc}(\mathbb{R}^2))$ is a weak solution of the vorticity formulation of the incompressible 2D Euler equations with initial data $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ if, for any test function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2)$, we have:*

$$\int_0^\infty \left(\int_{\mathbb{R}^2} \varphi_t \mathrm{d}\omega(x,t) + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi d\omega(x,t) \otimes \mathrm{d}\omega(y,t) \right) \mathrm{d}t + \int_{R^2} \varphi(x,0)\mathrm{d}\omega_0(x) =$$

$$\int_0^\infty \int_{\mathbb{R}^2} \nabla^\perp \varphi \cdot \mathrm{d}F(x,t)\mathrm{d}t,$$

where

$$H_\varphi(x,y,t) \equiv \frac{\nabla\varphi(x,t) - \nabla\varphi(y,t)}{4\pi|x-y|} \cdot \frac{(x-y)^\perp}{|x-y|}.$$

This formulation is equivalent to the one given in Definition 1, as long as the velocity is $(L^2_{\mathrm{loc}}(\mathbb{R}^2))^2$ (Schochet, 1995, Lemma 2.1). The original equivalence result was stated for the homogeneous system, but the proof carries over in a straightforward manner to the inhomogeneous case.

Finally, for $n \in \mathbb{N}$, we recall the notation

$$(2n)!! = \prod_{\ell=1}^n (2\ell), \quad \text{and} \quad (2n-1)!! = \prod_{\ell=1}^n (2\ell - 1).$$

3. The governing equation

In what follows, we shall show that the velocity associated to (2.2) solves, in a weak sense, the system of equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + F \\ \operatorname{div} u = 0, \end{cases} \tag{3.1}$$

with the forcing term

$$F(x_1, x_2) = -\pi/8 \begin{bmatrix} \delta(x_1 - 1, x_2 + t/2) - \delta(x_1 + 1, x_2 + t/2) \\ 0 \end{bmatrix}, \tag{3.2}$$

where δ represents the Dirac point mass at the origin in the plane.

More precisely, we have:

Theorem 3.1. *The steadily translating vorticity (2.2) is a weak solution of the vorticity formulation of the inhomogeneous incompressible 2D Euler equations, with forcing term (3.2).*

The multiplying constant in (3.2) should be compared with equation (7) in § 6.3 of Saffman (1992)†. Thus, in accordance with Saffman's analysis, the forcing given by (4) can be thought as a tension applied near the tips of the vortex sheet.

To present the proof of this result we need to prove first a pair of lemmas.

Lemma 3.2. *Set*

$$a_n = \frac{(2n+1)!!}{(2n+2)!!} + \frac{1}{2}\sum_{\ell=1}^{n} \frac{(2n+1-2\ell)!!}{(2n+2-2\ell)!!}\frac{(2\ell-1)!!}{(2\ell)!!},$$

for $n \geq 2$, and set $a_0 = a_1 = 1/2$. Then, for all $n = 2, 3, \ldots$, $a_n = 1/2$.

Proof. Write:

$$b_n = \frac{(2n+1)!!}{(2n+2)!!}.$$

Then, with this notation we have:

$$a_n = b_n + \frac{1}{2}\sum_{\ell=1}^{n} b_{n-\ell}b_{\ell-1},$$

so that what we want to show is that the right hand side of the equality above is identically equal to 1/2, for $n = 2, 3, \ldots$. We will use a generating function argument.

We introduce the function:

$$f(x) = \sum_{n=0}^{\infty} b_n x^n.$$

It can be easily checked that

$$f(x) = \frac{1}{x}\left(\frac{1}{\sqrt{1-x}} - 1\right).$$

The identity sought reduces, through term-by-term identification of the Taylor coefficients, to the following algebraic identity:

$$f(x) - \frac{1}{2} - \frac{3x}{8} + \frac{x}{2}((f(x))^2 - \frac{1}{4}) = \frac{1}{2}\left(\frac{1}{1-x} - 1 - x\right).$$

□

We introduce the compactly supported distribution S, which paired with any test function $\eta \in C^\infty(\mathbb{R})$ gives:

$$\langle S, \eta \rangle = \frac{1}{2}\int_{-1}^{1} \eta(x)\frac{x}{\sqrt{1-x^2}}\mathrm{d}x + \frac{1}{4\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{\eta(x)-\eta(y)}{x-y}\frac{x}{\sqrt{1-x^2}}\frac{y}{\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y.$$

† For comparison purposes, we observe that, in our case, we have $U = 1/2$ and $a = 1$.

Lemma 3.3. *The following identity holds:*

$$S = \frac{\pi}{8}(\delta_1 - \delta_{-1}),$$

where for any $a \in \mathbb{R}$, δ_a is the Dirac measure at the point a.

Proof. We start by observing that the support of S is obviously contained in the closed interval $[-1, 1]$.

Next we observe that S is an odd distribution, that is, if η is an even test function then $\langle S, \eta \rangle = 0$. Indeed, the first integral vanishes trivially, whereas to see that the second vanishes it suffices to use the change of variables $(x, y) \mapsto (-x, -y)$.

Now we show that $\langle S, \eta \rangle$ vanishes for any η which is a polynomial vanishing at $x = 1$ and $x = -1$ simultaneously. Clearly it is enough to verify this for all test functions of the form $\eta_n(x) = (1 - x^2)x^{2n-1}$.

First observe that, for all $k = 1, 2, \ldots$:

$$\int_{-1}^{1} \frac{x^{2k}}{\sqrt{1-x^2}} \mathrm{d}x = \int_0^{\pi} (\cos\theta)^{2k} \mathrm{d}\theta = \frac{(2k-1)!!}{(2k)!!}\pi,$$

where the second equality is a well-known calculus identity (see Ryzhik & Gradshteyn, 1994).

We compute:

$$\int_{-1}^{1} \eta_n(x) \frac{x}{\sqrt{1-x^2}} \mathrm{d}x = \pi \left(\frac{(2n-1)!!}{(2n)!!} - \frac{(2n+1)!!}{(2n+2)!!} \right); \tag{3.3}$$

$$\frac{\eta_n(x) - \eta_n(y)}{x - y} = \sum_{\ell=0}^{2n-2} x^{\ell} y^{2n-2-\ell} - \sum_{\ell=0}^{2n} x^{\ell} y^{2n-\ell}; \tag{3.4}$$

hence, using (3.4) in the double integral we obtain, after noticing that the even powers integrate to zero:

$$\int_{-1}^{1} \int_{-1}^{1} \frac{\eta_n(x) - \eta_n(y)}{x - y} \frac{x}{\sqrt{1-x^2}} \frac{y}{\sqrt{1-y^2}} \mathrm{d}x\mathrm{d}y =$$

$$\sum_{\ell=1}^{n-1} \int_{-1}^{1} \int_{-1}^{1} x^{2\ell-1} y^{2n-1-2\ell} \frac{x}{\sqrt{1-x^2}} \frac{y}{\sqrt{1-y^2}} \mathrm{d}x\mathrm{d}y +$$

$$- \sum_{\ell=1}^{n} \int_{-1}^{1} \int_{-1}^{1} x^{2\ell-1} y^{2n+1-2\ell} \frac{x}{\sqrt{1-x^2}} \frac{y}{\sqrt{1-y^2}} \mathrm{d}x\mathrm{d}y =$$

$$\pi^2 \left[\sum_{\ell=1}^{n-1} \frac{(2\ell-1)!!}{(2\ell)!!} \frac{(2n-2\ell-1)!!}{(2n-2\ell)!!} - \sum_{\ell=1}^{n} \frac{(2\ell-1)!!}{(2\ell)!!} \frac{(2n-2\ell+1)!!}{(2n-2\ell+2)!!} \right].$$

What we wish to prove is that

$$0 = \langle S, \eta_n \rangle = \frac{1}{2}\pi \left(\frac{(2n-1)!!}{(2n)!!} - \frac{(2n+1)!!}{(2n+2)!!} \right) +$$

$$\frac{1}{4\pi}\pi^2 \left[\sum_{\ell=1}^{n-1} \frac{(2\ell-1)!!}{(2\ell)!!} \frac{(2n-2\ell-1)!!}{(2n-2\ell)!!} - \sum_{\ell=1}^{n} \frac{(2\ell-1)!!}{(2\ell)!!} \frac{(2n-2\ell+1)!!}{(2n-2\ell+2)!!} \right].$$

In the notation of Lemma 3.2 this can be written as $a_{n-1} - a_n = 0$, which clearly holds.

The polynomials which vanish at the endpoints of the interval $[-1,1]$ are dense in $C_c^\infty((-1,1))$ and hence S vanishes against any such test function. It follows that the support of S is contained in the set $\{-1,1\}$. A distribution with support in a finite set of points is a finite sum of Diracs and its derivatives. Using the fact that S is odd, we write:

$$S = \sum_{k=0}^{N} a_k(\delta_1^{(k)} - (-1)^k \delta_{-1}^{(k)}).$$

If $N \geq 1$ we can use the polynomial $p(x) = x(1-x^2)^N$ as test function to show that $a_N = 0$. This implies that $S = a_0(\delta_1 - \delta_{-1})$.

Finally, considering $\eta(x) = x$ we conclude that $a_0 = \pi/8$, as we wished.

□

We now give the proof of our main result.

Proof of Theorem 1. We will reduce the proof to Lemma 3.3. Let us begin by choosing a test function of the form

$$\varphi(x_1, x_2, t) = \alpha(t)\zeta\left(x_1, x_2 + \frac{t}{2}\right),$$

where α and ζ are C_c^∞ in $[0,\infty)$ and $\mathbb{R}^2$, respectively. We then compute:

(*i*)

$$\int_0^\infty \int_{\mathbb{R}^2} \varphi_t \mathrm{d}\omega(x,t)\mathrm{d}t =$$

$$\int_0^\infty \int_{-1}^1 \left(\alpha'(t)\zeta(x_1,0) + \alpha(t)\frac{1}{2}\zeta_{x_2}(x_1,0) \right) \frac{x_1}{\sqrt{1-x_1^2}} \mathrm{d}x_1 \mathrm{d}t =$$

$$-\int_{-1}^1 \alpha(0)\zeta(x_1,0)\frac{x_1}{\sqrt{1-x_1^2}} dx_1 + \int_0^\infty \alpha(t)\frac{1}{2}\int_{-1}^1 \zeta_{x_2}(x_1,0)\frac{x_1}{\sqrt{1-x_1^2}} \mathrm{d}x_1 \mathrm{d}t;$$

(ii)

$$\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi \mathrm{d}\omega(x,t)\otimes d\omega(y,t)\mathrm{d}t =$$

$$\int_0^\infty \alpha(t)\frac{1}{4\pi}\int_{-1}^1\int_{-1}^1 \frac{\zeta_{x_2}(x_1,0)-\zeta_{x_2}(y_1,0)}{x_1-y_1}\frac{x_1}{\sqrt{1-x_1^2}}\frac{y_1}{\sqrt{1-y_1^2}}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}t;$$

(iii)

$$\int_{R^2}\varphi(x,0)\mathrm{d}\omega_0(x) = \int_{-1}^1 \alpha(0)\zeta(x_1,0)\frac{x_1}{\sqrt{1-x_1^2}}\mathrm{d}x_1.$$

Therefore, adding the three terms and using the notation of Lemma 3.3 we obtain:

$$\int_0^\infty \left(\int_{\mathbb{R}^2}\varphi_t\mathrm{d}\omega(x,t) + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi\mathrm{d}\omega(x,t)\otimes\mathrm{d}\omega(y,t)\right)\mathrm{d}t + \int_{R^2}\varphi(x,0)\mathrm{d}\omega_0(x) =$$

$$\int_0^\infty \alpha(t)\langle S,\zeta_{x_2}(\cdot,0)\rangle\mathrm{d}t =$$

$$\int_0^\infty \alpha(t)\left(\frac{\pi}{8}\zeta_{x_2}(1,0)-\frac{\pi}{8}\zeta_{x_2}(-1,0)\right)\mathrm{d}t = \int_0^\infty\int_{\mathbb{R}^2}\nabla^\perp\varphi\cdot\mathrm{d}F(x,t)\mathrm{d}t,$$

as we wished. We then observe that finite sums of test functions having the form we used are dense in the set of all $C_c^\infty([0,\infty)\times\mathbb{R}^2)$ test functions.

□

4. Conclusions

Our result is a rigorous confirmation of the conclusion obtained from a fluid dynamical standpoint by (Saffman, 1992, § 6.2). The need for an external force to keep the Prandtl-Munk vortex sheet from rolling-up is due to the rather strong singularity of the vorticity at the tips.

Another example of vortex sheet generating a flow which keeps a constant shape vorticity profile is

$$\omega_0 = \chi_{[-1,1]}(x_1)\sqrt{1-x_1^2}\otimes\delta_0(x_2),$$

which rotates with constant angular velocity 1 about the origin (see Saffman, 1992, § 9.3). Saffman argues that the singularity at the tips of the vortex sheet in this case is not strong enough to produce a force imbalance, so that this steadily rotating vortex sheet generates a physically correct flow. This can also be confirmed rigorously, since this vorticity configuration is a limit of a sequence of Kirchhoff ellipses, which are known steadily rotating weak solutions of the incompressible 2D Euler equations (again see Saffman, 1992, § 9.3). Hence, the weak limit of such a sequence of approximations must also be a weak solution, which follows by applying any of a number of results, see for example, DiPerna & Majda (1987); Delort (1991); Lopes Filho *et al.* (2000). We also observe that a tip singularity is not an essential condition for roll-up of vortex sheets with finite length: Schwartz (1981) presents calculations with vortex sheets without singularities that evolve to a roll-up.

Further, the result presented here raises a number of interesting issues. First, it rules out the Prandtl-Munk vortex sheet as a natural example of non-uniqueness of weak solutions to the Euler equations. Recent numerical evidence of such non-uniqueness has been found by Lopes Filho *et al.* (1999). Also, the implications of this result to calculations of vorticity distributions that minimise the induced drag on a wing (for instance Munk, 1919) are not entirely clear.

Finally, this results also suggests that not every solution to the Birkhoff-Rott equation is a solution (even in the weak sense) to the Euler equations, contrary to an apparently common wisdom that the Birkhoff-Rott equation would give a full description of the evolution of a vortex sheet (see for instance Saffman, 1992, pg. 142). In particular, we observe that, as far as the Birkhoff-Rott evolution is concerned, if we exclude the points $z = \pm 1$ from the domain, then the translating Prandtl-Munk vortex sheet is a *bona-fide* solution. While such a solution is probably not relevant for the physical problem, it raises the question of what are the classes of solutions for which there is an equivalence between the Euler and the Birkhoff-Rott equations.

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References

BATCHELOR, G. K. 1967 *An introduction to fluid dynamics*. Cambridge University Press.

DELORT, J.-M. 1991 Existence de nappes de tourbillon en dimension deux. *J. of Amer. Math. Soc.* **4**, 553–586.

DIPERNA, R. & MAJDA, A. 1987 Concentrations and regularizations for 2d incompressible flow. *Comm. Pure and Appl. Math* **XL**, 301–345.

KLEIN, F. 1910 Uber bie Bildung von Wirbeln in reibungslosen Flüssigkeiten. *Zeit. für Math. u. Physik* **59**, 259–262.

KRASNY, R. 1987 Computation of vortex sheet roll-up in Trefftz plane. *J. Fluid Mech.* **184**, 123–155.

LOPES FILHO, M. C., LOWENGRUB, J., NUSSENZVEIG LOPES, H. J. & ZHENG, Y. 1999 Numerical evidence for the nonunique evolution of vortex sheets in the plane. Submitted.

LOPES FILHO, M. C., NUSSENZVEIG LOPES, H. J. & TADMOR, E. 2000 Approximate solutions of the incompressible euler equations with no concentrations. To appear in Ann. Inst. H. Poincaré C - An. non-Lineaire.

MARCHIORO, C. & PULVIRENTI, M. 1994 *Mathematical Theory of Incompressible Nonviscous Fluids*. Springer-Verlar.

MEYER, R. K. 1982 *Introduction to Mathematical Fluid Dynamics*. Dover.

MOORE, D. W. 1976 The stability of an evolving two-dimensional vortex sheet. *Mathematika* **23**, 35–44.

MUNK, M. 1919 Isoperimetrische Aufgaben aus der Theorie des Fluges. Inaug.-dissertation, Göttingen.

RYZHIK, I. M. & GRADSHTEYN, I. S. 1994 *Table of integrals, series and products*. Academic Press.

SAFFMAN, P. G. 1992 *Vortex Dynamics*. Cambridge University Press.

SCHOCHET, S. 1995 The weak vorticity formulation of the 2d euler equations and concentration-cancellation. *Comm P.D.E.* **20**.

SCHWARTZ, L. W. 1981 A semi-analytic approach to the self-induced motion of vortex sheets. *J. Fluid Mech.* **111**, 475–490.