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On canonical curves and osculating spaces ☆

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Abstract

We study the geometry of a reduced canonical curve with a nondegenerate component. We prove that the other components are rational normal curves in a certain configuration. In addition, given a nonsingular point on a nondegenerate component, we analyze the relationship between the Weierstrass semigroup and the intersection divisors of the osculating spaces with the curve. We describe how these divisors vary and present an upper bound for their degrees. We study in detail the curves that attain this bound. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let N be a numerical semigroup, that is, a subsemigroup of the natural numbers whose complement is finite, say with g elements. Pinkham [14] constructed a moduli space of pointed smooth curves of genus g whose Weierstrass semigroup at the distinguished point is N. Using a variant of Petri's analysis, Stöhr [17] and Oliveira and Stöhr [12] constructed a compactification for the Pinkham's space for semigroups whose last gap is $2g-1$ or $2g-2$, respectively. In that case, the boundary consists of reduced Gorenstein curves that, when canonically embedded, have a nondegenerate component. Our aim is to show how to apply that construction further.

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Let C be a nonhyperelliptic smooth curve of genus g. Then the canonical map, given by the regular differentials on C, is an embedding. The image is nondegenerate in $\mathbb{P}^{g-1}$ (that is, not contained in any hyperplane) and of degree $2g-2$. Recall that the *Weierstrass semigroup* of a point P of C is the set of pole numbers of rational functions that are regular away from P. Its complement in the natural numbers consists of g elements, say $\ell_1 < \ell_2 < \cdots < \ell_g$, called its *gaps*. For $i = 0, \ldots, g-2$, the intersection of all hyperplanes of $\mathbb{P}^{g-1}$ that meet C at P with order at least $\ell_{i+2} - 1$ is a subspace of dimension i, called the *ith osculating space* of C at P.

We relate the gaps and the osculating spaces as follows. Let $E_{\ell_{i+2}}$ be the intersection divisor of the ith osculating space and the curve outside P (see Section 4). Pimentel [13, Theorem 1.1] proved that the sequence $E_{\ell_2} \leqslant E_{\ell_3} \leqslant \cdots \leqslant E_{\ell_g}$ can vary only at indices of a certain subset of the gaps. We improve his result in Theorem 4.1, proving that subset can be restricted to the *translation* gaps ℓ (that is, $\ell + n$ is a nongap for every positive nongap n). In Corollary 4.3, we show that if the number of translation gaps, called the *type*, is at most two, then a divisor E_ℓ is either zero or is equal to E_{ℓ_g}. In that case [13, Theorem 2.1], there is a certain basis for the spaces of the higher-order differentials and the Oliveira and Stöhr construction applies.

To deal with types greater than two, we stratify the moduli space according to the behavior of the osculating spaces. In Theorem 4.5, we prove an upper bound on the degree of divisors E_ℓ. In Section 5, we study the curves that attain that bound. In this case, the type of the Weierstrass semigroup at P is maximal (see Definition 2.3). In (5.2) and (5.3), we present examples of such curves. Furthermore, we show in Example 5.2 that, in general, the Weierstrass semigroup does not determine the behavior of osculating spaces. In Theorem 5.5, we present a basis for the spaces of the higher-order differentials on such curves, and conclude that they are arithmetically Cohen–Macaulay.

Even if we are only interested in smooth curves, we are led naturally to consider certain singular Gorenstein curves too, since they appear in the boundary of the moduli space. More precisely, we consider connected, reduced, canonically embedded curves, with a nondegenerate component. In Section 3, we characterize such curves. In Theorem 3.1, we show that each other component is a rational normal curve in the subspace that it spans. The results stated above hold for such curves, provided that the osculating hyperplane meets the curve only at nonsingular points.

In addition, we prove the following results. In Section 2, we deal with numerical semigroups. In Proposition 2.1, we prove that every semigroup has a set of generators determined by its translation gaps. In Proposition 2.5, we prove a lower bound on the cardinality of sets of sums of gaps and show that this bound is achieved precisely by the semigroups whose type reaches its maximal value.

In Section 6, we realize a certain family of semigroups of maximal type using reducible curves. As a byproduct, we obtain certain Gorenstein curves containing an irreducible component that is not Gorenstein, generalizing [11, Section 4]. In contrast, we show in Proposition 6.5 that some semigroups of our family cannot be realized as Weierstrass semigroups of a smooth curve.

Below, by a *semigroup* we mean a numerical semigroup. All curves that we consider are complete, reduced algebraic schemes of pure dimension one, defined over an algebraically closed field K.

2. Numerical semigroups

Let N be a semigroup and g its *genus*, that is, the number of its gaps. We write $L=\{\ell_1<\ell_2<\cdots<\ell_g\}$ for the set of gaps. Set $d{:=}2g-1-\ell_g$. The last gap satisfies $g\leqslant\ell_g\leqslant 2g-1$, and so we have $0\leqslant d\leqslant g-1$. A gap ℓ is called *special* if $\ell_g-\ell$ is a gap. Since there exist $g-1-d$ positive nongaps smaller than the last gap, there are exactly d special gaps, say $s_1<s_2<\cdots<s_d$.

The positive integers t such that $t+n$ is a nongap for every positive nongap n are called the *translations* of N (cf. [15, p. 457]). The *type* τ of N is the number of its translation gaps. We denote the translation gaps by

$$T{:=}\{t_1<t_2<\cdots<t_\tau\}.$$

Note that the last gap is a translation and every other translation gap is special and so $1\leqslant\tau\leqslant d+1$.

Below, we prove that every semigroup has a set of generators determined by its translations gaps.

Proposition 2.1. *Let n_1 be the smallest positive nongap of N. Then the following statements hold.*

(i) *The semigroup N is generated by*

$$\{n\in N\mid n<\ell_g\}\cup\{n_1+t\mid t\in T\}.$$

If $n_1>d+2$, then $n_1+\ell_g$ can be excluded from this set.

(ii) *We have $n_1\geqslant\tau+1$. If the equality holds, then $\{n_1,n_1+t_1,n_1+t_2,\ldots,n_1+t_\tau\}$ is a set of generators for N.*

(iii) *The set of translations of N is a semigroup generated by $\{n\in N\mid n<\ell_g\}\cup T$.*

Proof. (i) We must prove that each integer n satisfying $\ell_g+1\leqslant n\leqslant\ell_g+n_1$ can be written as a linear combination of the claimed generators. If $\ell_g-n_1<n-n_1<\ell_g$ then $n-n_1\in N$ or $n-n_1\in T$, since every gap greater than ℓ_g-n_1 is a translation. Now, if $n=n_1+\ell_g$ and n_1 is greater than $d+2$, then the number of pairs $(r,n-r)$ with $n_1+1\leqslant r\leqslant\ell_g-1=2g-2-d$ is greater than the number $2(g-n_1)$ of gaps appearing in these pairs, and so at least one such pair contains two nongaps.

(ii) Being translation gaps, the residual classes of $t_1,t_2,\ldots,t_\tau$ modulo n_1 are nonzero and pairwise different and so $n_1\geqslant\tau+1$. Suppose $n_1=\tau+1$. Let n be a nongap and t a translation gap with same residual classes modulo n_1. Then $n>t$ and so n is a linear combination of n_1 and t.

(iii) It is clear that the translations form a semigroup. The assertion about its generators follows directly from Statement (i). □

Note that if $t_1 = 1$, then $n + 1$ is a nongap for every positive nongap n and so the gap sequence is $1, \dots, g$.

Theorem 2.2. *Suppose the gap sequence of* N *differs from* $1, \dots, g$. *Then we can write each* $r \in \{2, \dots, \ell_g - 1\}$ *as a sum of gaps* a, b, *in which at least one of them is not a translation.*

Proof. The existence of a partition of r as a sum of two gaps follows from [10, Theorem 1.3]. We may assume $r > t_1$. Since $L \neq \{1, \dots, g\}$, we have $t_1 > 1$ and so we may assume $r - 1$ is a nongap.

Let Γ denote the set of translations. We consider two cases.

Suppose $r \notin \Gamma$. Then r is a gap. So $r - t_1$ is a gap as well and, since Γ is a semigroup, it is not a translation. We are done in this case.

Suppose $r \in \Gamma$. Assume, by way of contradiction, that whenever $r = a + b$ is a sum of gaps, then a, b are translations. Let

$$\tilde{N} := \{0\} \cup \{n \in N \mid r - n \in L\} \cup \{r + 1, r + 2, \dots\}.$$

It is easy to check that $\tilde{N}$ is a semigroup whose last gap is r. Let $\tilde{n}_1$ be its smallest positive nongap and $\tilde{S}$ (resp. $\tilde{T}$) its set of special (resp. translation) gaps. If $\tilde{s}$ is a special gap of $\tilde{N}$, then

$$\tilde{s} \in L \quad \text{if and only if } r - \tilde{s} \in L$$

and so by our contradiction hypothesis we have $\tilde{S} \subset \Gamma$. We conclude that $\tilde{T} \subset \Gamma$, since $\tilde{T} \subset (\tilde{S} \cup \{r\})$. On the other hand, it follows from Proposition 2.1 that

$$\tilde{G} := \{\tilde{n} \in \tilde{N} \mid \tilde{n} < r\} \cup \{\tilde{n}_1 + \tilde{t} \mid \tilde{t} \in \tilde{T}\}$$

is a set of generators for $\tilde{N}$. Since $\tilde{n}_1 \in N$ (as $r - 1$ is a nongap), we obtain $\tilde{G} \subset N$, that is, $\tilde{N} \subset N$, which is a contradiction, because $\ell_g \in \tilde{N}$. The theorem is proved. □

Definition 2.3. A semigroup is said to be of *maximal type* if each special gap is a translation, or equivalently, if $\tau = d + 1$.

We shall meet semigroups of maximal type in Section 5, as Weierstrass semigroups of pointed curves whose osculating spaces intersect the curve "maximally". Semigroups of maximal type also appear as semigroups of values of certain one-dimensional local rings and were named *almost symmetric* by Barucci and Fröberg; see [2] for further information.

Remark 2.4. Let s be a special gap and n be a positive nongap. If $s + n$ is a gap, then it must be a special gap and so

$$s \in T \quad \Leftrightarrow \quad s' - s \text{ is a gap for every special gap } s' > s.$$

In particular, N is of maximal type if and only if $s_j - s_i$ is a gap whenever $j > i$. Note that if $d = 0$, then the semigroup has type one. Conversely, if $d \geqslant 1$, then the last special gap is a translation and so the type is at least two.

A semigroup is called *symmetric* if $\ell_g - \ell$ is a nongap for each gap ℓ, that is, if $d = 0$. By Remark 2.4, a semigroup is symmetric if and only if it has type one. We say that a semigroup is *quasi-symmetric* if it has only one special gap, namely $g - 1$. Quasi-symmetric semigroups have type two and are also of maximal type.

For each $n \geqslant 2$, let

$$L_n := \left\{ \sum_{i=1}^{n} a_i \,|\, a_1, \ldots, a_n \in L \right\}$$

be the set of sums of n gaps. Oliveira [10, Theorem 1.5] proved that a semigroup with $\ell_2 = 2$ is symmetric if and only if $\#L_n = (2n-1)(g-1)$ for every $n \geqslant 2$. Below, we give a lower bound on the cardinality of L_n for nonsymmetric semigroups. The semigroups that attain this bound are exactly those with maximal type.

Proposition 2.5. *Let N be a nonsymmetric semigroup. For each $n \geqslant 2$, set*

$$M_n := \{n, \ldots, (n-1)\ell_g\} \cup \{(n-1)\ell_g + \ell \,|\, \ell \in L\}.$$

The following statements hold:

(i) $M_n \subseteq L_n$.

(ii) $(2n-1)(g-1) - (n-1)d + 1 \leqslant \#L_n \leqslant (2n-1)(g-1) - nd + g$.

(iii) *The following five assertions are equivalent*:

(a) *The semigroup N is of maximal type.*

(b) $L_n = M_n$ *for some* $n \geqslant 2$.

(c) $L_n = M_n$ *for every* $n \geqslant 2$.

(d) $\#L_n = (2n-1)(g-1) - (n-1)d + 1$ *for some* $n \geqslant 2$.

(e) $\#L_n = (2n-1)(g-1) - (n-1)d + 1$ *for every* $n \geqslant 2$.

Proof. Statement (i) follows from the proof of [11, Theorem 1.1], but we repeat the argument here for the reader's convenience. Let $n \geqslant 2$ and $m \in \{n, \ldots, (n-1)\ell_g\}$. We write $m - n = i(\ell_g - 1) + (r - 2)$ in a such way that $0 \leqslant i \leqslant n-2$ and $2 \leqslant r \leqslant \ell_g$. Thus, $m = i\ell_g + (n-2-i)\ell_1 + r$ and so the inclusion $M_n \subseteq L_n$ holds, since each $r \in \{2, \ldots, \ell_g\}$ can be written as a sum of two gaps whenever N is not symmetric (cf. [10, Theorem 1.3]). Now Statement (ii) follows by observing that $\#M_n = (2n-1)(g-1) - (n-1)d + 1$ and $L_n \subseteq \{n, n+1, \ldots, n\ell_g\}$.

Let us prove Statement (iii). We have (c) implies (b) and, by Statement (i), (b) is equivalent to (d) and (c) is equivalent to (e).

(b) $\Rightarrow$ (a): suppose $L_n = M_n$ and let s be a special gap. If h is a positive nongap, then $(n-1)\ell_g + h$ does not belong to L_n. So $(n-2)\ell_g + (\ell_g - s) + (s + h)$ is not a sum of gaps, that is, $s + h \in N$. Thus, each special gap is a translation and so N has maximal type.

(a) $\Rightarrow$ (c): let $a_1,\ldots,a_n$ be gaps and $0<r<\ell_g$ be such that $\sum_{i=1}^n a_i=(n-1)\ell_g+r$. Note that ℓ_g-a_i is a translation for each i: indeed, either ℓ_g-a_i is a nongap or it is a special gap (hence a translation, as N is of maximal type). By writing $a_n=\sum_{i=1}^{n-1}(\ell_g-a_i)+r$, we conclude r is a gap. We proved $L_n\subseteq M_n$ and now the result follows from Statement (i). $\square$

3. Canonical curves with a nondegenerate component

The intrinsic geometry of an abstract curve can be translated in terms of the projective geometry of its canonical model. For singular curves, the canonical map is obtained by replacing the canonical sheaf by the *dualizing sheaf*, whose global sections we now describe.

Let C be a curve and $C_1,\ldots,C_n$ its irreducible components. Let $v:\tilde{C}\to C$ be its normalization. The curve $\tilde{C}$ is the disjoint union of normalizations $\tilde{C}_1,\ldots,\tilde{C}_n$ of the components. The space of *rational differentials* on $\tilde{C}$ is the direct product

$$\Omega^1_{\tilde{C}}=\Omega^1_{\tilde{C}_1}\times\cdots\times\Omega^1_{\tilde{C}_n}$$

of rational differentials on each component. We say that $\omega\in\Omega^1_{\tilde{C}}$ is a *regular differential on C* if for each $P\in C$ and for each regular function z of C at P we have

$$\sum_{Q\in v^{-1}(P)}\mathrm{res}_Q(v^*z\cdot\omega)=0. \tag{1}$$

We identify the global sections of the dualizing sheaf ω_C of C with the regular differentials on C (cf. [6, p. 82]). In order to define a map of the regular differentials to some projective space, the dualizing sheaf must be invertible, or equivalently, the curve C must be Gorenstein. If g is the arithmetic genus of C, then by definition

$$g=1+h^1(C,\mathcal{O}_C)-h^0(C,\mathcal{O}_C). \tag{2}$$

By duality, $h^0(C,\omega_C)=h^1(C,\mathcal{O}_C)$, and so the dimension of the space of regular differentials coincides with the arithmetic genus if and only if C is connected. Given a regular differential ω on C, we write

$$\mathrm{div}(\omega):=\sum_{P\in C_{\mathrm{reg}}}\mathrm{ord}_P(\omega)\cdot P$$

for the corresponding Weil divisor supported on nonsingular points of C. If D is a Weil divisor supported on C_{reg}, then we shall, by abuse of notation, write

$$\Omega^n_C(D)=H^0(C,\omega_C^{\otimes n}(-D))$$

to denote the space of n-fold regular differentials φ such that $\mathrm{div}(\varphi)\geqslant D$.

From now on, we assume that C is a connected Gorenstein curve of arithmetic genus g and the canonical map is an embedding. We identify C with its image, thus obtaining a nondegenerate curve in $\mathbb{P}^{g-1}$ of degree $2g-2$.

To any curve Z we associate a *graph*, defined as follows: to each component Y of Z, take a segment and mark on it one point for each singular point of Z that belongs to Y. Finally, identify the points belonging to more than one component. If Z is a subcurve of C, we denote by Z^c the subcurve of C given by the union of components not contained in Z.

Theorem 3.1. *Let X be a connected subcurve of C. Suppose X is nondegenerate in $\mathbb{P}^{g-1}$. Then the following statements hold*:

(i) *Each connected subcurve of X^c has arithmetic genus zero and its singularities are given by the intersection of smooth branches with independent tangents.*

(ii) *The singularities of C outside X are ordinary nodes.*

(iii) *Each irreducible component $Y \subset X^c$ is a rational normal curve* (*in the subspace that it spans*). *In addition, the subcurve Y^c is connected.*

Proof. To begin with, let Z be any subcurve of C. Then the regular differentials on Z are precisely those given by restriction of regular differentials on C that vanish on Z^c. Indeed, let $Z = C_1 \cup \cdots \cup C_m$. By (1), we have that $(\omega_1, \ldots, \omega_m) \in \Omega^1_{\bar{Z}}$ is a regular differential on Z if and only if $(\omega_1, \ldots, \omega_m, 0, \ldots, 0)$ is regular on C.

Let Z be a connected subcurve of X^c. As X is nondegenerate, it follows that there does not exist a nonzero regular differential on C vanishing on X. So, by our remark above, the space of regular differentials on Z is the null space. Since Z is connected, it follows from (2) that its arithmetic genus is zero. Now the assertion about the singularities and the graph of Z follows from [3, Proposition 1.8]. Statement (i) is proved. Since the curve C is Gorenstein, the number of branches centered on a singular point of $C - X$ must be two and so Statement (ii) follows as well.

Let us prove Statement (iii). Let Y be an irreducible component of X^c. By Statement (i), the curve Y has arithmetic genus zero, so it is isomorphic to a projective line.

Claim. *The component Y is a rational normal curve if and only if Y^c is connected.*

Proof (*claim*). Since $Y \subset C$ are both Gorenstein curves, from [3, Lemma 1.12] we obtain $\deg(\omega_C|_Y) = \deg(\omega_Y) + \delta$, where δ is the length of $\mathcal{O}_{Y \cap Y^c}$. So, $Y \subset \mathbb{P}^{g-1}$ is an irreducible curve of degree $\delta - 2$. Let r be the dimension of the space of the regular differentials on Y^c. From (2), we have $r = g' - 1 + h^0(Y^c, \mathcal{O}_{Y^c})$, where g' is the arithmetic genus of Y^c. By our remark at the beginning of the proof, the curve Y spans a subspace of dimension $g - r - 1$. On the other hand, the arithmetic genus of C is $g = g' + \delta - 1$ (cf. [7, Theorem 3]) and so

$$g - r - 1 = \delta - 2 - (h^0(Y^c, \mathcal{O}_{Y^c}) - 1).$$

Hence, Y is a rational normal curve if and only if $h^0(Y^c, \mathcal{O}_{Y^c}) = 1$, proving the claim. □

To finish the proof of Statement (iii), we need only to prove that Y^c is connected. Let Z be a maximal connected subcurve of Y^c. It suffices to prove that Z contains the subcurve X. Suppose that it is not so. Then it follows from Statements (i) and (ii) that the graph of $Z \cup Y$ is contractible and its singularities outside X are ordinary nodes. Hence, there is a component $E \subset Z$ such that E meets the subcurve $\overline{(Z \cup Y) - E}$ just at one node. On the other hand, since the map induced by $\omega_C|_E$ is an embedding and E is smooth and rational, the component E intersects its complement E^c at least at three points, counting the multiplicities. Therefore, E intersects the subcurve $(Z \cup Y)^c$, reaching a contradiction. The theorem is proved. □

Remark 3.2. Suppose C has two distinct nondegenerate irreducible components, say X and Y. Then their degrees are at least $g-1$. Since C has degree $2g-2$, we conclude that X and Y have degree $g-1$ and $C = X \cup Y$. So, X and Y are rational normal curves and therefore each nonsingular point of C is noninflectionary. In characteristic zero, Garcia and Lax [4] have shown that for every odd number $n \geqslant 3$, there exists an *irreducible* rational curve with n ordinary nodes such that every nonsingular point is a nonWeierstrass point.

4. Osculating spaces and Weierstrass semigroups

As before, let C be a canonically embedded Gorenstein curve of arithmetic genus g. Let P be a nonsingular point of C and X the irreducible component of C containing P. Let L be the set of integers ℓ such that there exists a hyperplane $H \subset \mathbb{P}^{g-1}$, not containing X, such that $\mathrm{ord}_P(X.H) = \ell - 1$ (that is, there exists a regular differential on C vanishing with order $\ell - 1$ at P). When C is smooth, L is the set of *Weierstrass gaps*. To establish a common nomenclature with the smooth case, we call the elements of L the *gaps* of C at P. The number of gaps is the dimension of the subspace spanned by X plus one and so is, at most, g. We assume that there are g gaps at P or equivalently, the component X is nondegenerate. In addition, we assume that the complement of the gaps in the natural numbers is a semigroup and call it *Weierstrass semigroup* of C at P.

Let $\ell_1 < \ell_2 < \cdots < \ell_g$ be the gaps. Let $T^{(i)}$ be the intersection of all hyperplanes of $\mathbb{P}^{g-1}$ that meet C at P with multiplicity at least $\ell_{i+2} - 1$. Each $T^{(i)}$ is a projective subspace of dimension i, called the *ith-osculating space* of C at P. So, $T^{(0)}$ is the point P, $T^{(1)}$ is the tangent line and $T^{(g-2)}$ is the *osculating hyperplane* of C at P. Put $d := \deg(C^*.T^{(g-2)})$, where C^* is the punctured curve $C \setminus \{P\}$. The contact order of the osculating hyperplane and the curve at P is $\ell_g - 1$ and so $\ell_g = 2g - 1 - d$. We assume that the osculating hyperplane meets the curve C outside P at least at one point, that is, $d \geqslant 1$. In addition, we assume that $C.T^{(g-2)}$ consists only of nonsingular points of C.

When $T \subset \mathbb{P}^{g-1}$ is a projective subspace of codimension possibly greater than one, we define the intersection divisor $C.T$ as the infimum of divisors $C.H$, where H is a hyperplane containing T. Thus, for $i = 0, \ldots, g-2$,

$$C.T^{(i)} = \sum_{Q \in C \cap T^{(g-2)}} \min\{\mathrm{ord}_Q(C.H) \mid H \text{ is a hyperplane}, \mathrm{ord}_P(C.H) \geqslant \ell_{i+2} - 1\} \cdot Q$$

or equivalently,

$$C.T^{(i)} = \sum_{Q \in C \cap T^{(g-2)}} \min\{\mathrm{ord}_Q(\omega) \mid \omega \in \Omega^1_C((\ell_{i+2} - 1)P)\} \cdot Q.$$

We denote by ω_ℓ a regular differential on C whose order at P is $\ell - 1$. Given a vector space of regular differentials, a basis for it is called *P-hermitian* if the orders at P of its elements are pairwise distinct.

Let $\{\omega_\ell \mid \ell \in L\}$ be a P-hermitian basis for the space of regular differentials on C. Note that $\mathrm{div}(\omega_{\ell_g})$ does not depend on the choice of the P-hermitian basis and it is equal to $C.T^{(g-2)}$. More generally, we have

$$C^*.T^{(i)} = \sum_{Q \in C^* \cap T^{(g-2)}} \min\{\mathrm{ord}_Q(\omega_\ell) \mid \ell \geqslant \ell_{i+2}\} \cdot Q \quad (i = 0, \ldots, g-2). \tag{3}$$

Put $E_{\ell_{i+2}} := C^*.T^{(i)}$ for each i. When C is smooth, Pimentel showed that the sequence $E_{\ell_2} \leqslant E_{\ell_3} \leqslant \cdots \leqslant E_{\ell_g}$ can change only at indices given by special gaps. More precisely, he proved [13, Theorem 1.1] that *if* $E_{\ell_i} < E_{\ell_{i+1}}$, *then* ℓ_i *is special.* We improve his result as follows.

Theorem 4.1. *If* $E_{\ell_i} < E_{\ell_{i+1}}$, *then* ℓ_i *is a translation gap.*

Proof. Assume ℓ_i is not a translation gap. Thus, there is a positive nongap n such that $\ell_i + n$ is a gap. Let $\{\omega_\ell \mid \ell \in L\}$ be a basis for the regular differentials on C. By [13, Proposition 1.6], we can assume, after an eventual change of basis, that $\mathrm{div}(\omega_\ell) \geqslant E_{\ell_g}$ for nonspecial gaps ℓ (the proof there, for smooth curves, works also in our case, since we are assuming the osculating hyperplane cuts C only at nonsingular points). Therefore, we can assume that ℓ_i is special (otherwise $E_{\ell_i} = E_{\ell_{i+1}}$), and in particular that $\ell_i + n \neq \ell_g$. Furthermore, we have $\mathrm{div}(\omega_{\ell_g - n}) \geqslant E_{\ell_g}$, since $\ell_g - n$ is nonspecial.

By the Riemann–Roch theorem, the vector space $\Omega^2_C(\mathrm{div}(\omega_{\ell_g}))$ has dimension g and so $\{\omega_\ell \omega_{\ell_g} \mid \ell \in L\}$ form a P-hermitian basis for it. Now, the sets

$$W_1 := \{\omega_\ell \omega_{\ell_g} \mid \ell > \ell_i\} \cup \{\omega_{\ell_i+n} \omega_{\ell_g - n}\} \quad \text{and} \quad W_2 := \{\omega_\ell \omega_{\ell_g} \mid \ell \geqslant \ell_i\}$$

span the same subspace of $\Omega^2_C(\mathrm{div}(\omega_{\ell_g}))$. Hence

$$\sum_{Q \in C^* \cap T^{(g-2)}} \min\{\mathrm{ord}_Q(\varphi) \mid \varphi \in W_1\} \cdot Q = \sum_{Q \in C^* \cap T^{(g-2)}} \min\{\mathrm{ord}_Q(\varphi) \mid \varphi \in W_2\} \cdot Q,$$

that is,

$$E_{\ell_i + n} + E_{\ell_g} = E_{\ell_i} + E_{\ell_g}$$

reaching a contradiction. The theorem is proved. □

Let $t_1 < t_2 < \cdots < t_\tau = \ell_g$ be the translation gaps of C at P.

Definition 4.2. The ascending sequence $E_{t_1}, E_{t_2}, \ldots, E_{t_\tau}$ is called the *sequence of* (*canonical*) *osculating divisors of C at P*.

As we have seen in Remark 2.4, a semigroup has type one if and only if every gap is nonspecial, that is, $d=0$. So, if the type of Weierstrass semigroup is one then the osculating hyperplane does not cut the curve outside P. In this case, $E_\ell=0$ for every gap ℓ. For semigroups of type two, we have the following result, which is a generalization of [13, Proposition 2.4].

Corollary 4.3. *Suppose the type of Weierstrass semigroup of C at P is two and let s be its last special gap. Then for each gap ℓ, we have $E_\ell=0$ if $\ell \leqslant s$ and $E_\ell=E_{\ell_g}$ otherwise.*

Proof. It follows from Theorem 4.1 that $E_\ell=E_{\ell_g}$ for each gap $\ell > t_{\tau-1}$ and $E_{t_1}=0$, since $E_{\ell_2}=0$. For semigroups whose type is greater than one, we have $s=t_{\tau-1}$. Since the type is two, the result follows. □

Lemma 4.4 (Pimentel [13, Lemma 1.3]). *Let ℓ,ℓ' be gaps such that $\ell+\ell'=\ell_g$. Then*

$$\operatorname{div}(\omega_\ell)+\operatorname{div}(\omega_{\ell'}) \not\geqslant E_{\ell_g}.$$

Proof. By the Riemann–Roch theorem, the vector space $\Omega_C^2(\operatorname{div}(\omega_{\ell_g})-P)$ has dimension g. Hence, the quadratic differentials $\{\omega_{\ell_i}\omega_{\ell_g} \mid i=1,\ldots,g\}$ form a P-hermitian basis for it. Since $\ell+\ell'$ does not belong to ℓ_g+L, we obtain that $\omega_\ell\omega_{\ell'}$ does not belong to $\Omega_C^2(\operatorname{div}(\omega_{\ell_g})-P)$ and so the lemma follows. □

Let $s_1<s_2<\cdots<s_d$ be the special gaps of C at P. Now, we prove an upper bound on the degree of divisors E_ℓ.

Theorem 4.5. *For each $i=1,\ldots,d$, we have* $\deg(E_{s_i}) \leqslant i-1$.

Proof. For $i=d$, by applying Lemma 4.4, we obtain

$$\operatorname{div}(\omega_{s_1})+\operatorname{div}(\omega_{s_d}) \not\geqslant E_{\ell_g}$$

and so $\deg(E_{s_d}) \leqslant d-1$. The general case follows by a similar argument. Indeed, let $D_j{:=}E_{\ell_g}-E_{s_{d+1-j}}$ for $j=1,\ldots,d$. So, for a given i, we must prove that $\deg(D_i) \geqslant i$. Consider the following set of i differentials

$$W{:=}\{\omega_{s_1},\omega_{s_2},\ldots,\omega_{s_i}\}.$$

By Lemma 4.4, we have $\operatorname{div}(\omega_{s_j}) \not\geqslant D_j$ for each j. Since $D_j \leqslant D_i$ whenever $j \leqslant i$, we obtain $\operatorname{div}(\omega) \not\geqslant D_i$ for each $\omega \in W$. Suppose now that $\deg(D_i)<i$. Write $D_i{:=}\sum_k a_k Q_k$ with each a_k positive. By doing a normalization (that is, a change of the P-hermitian basis), we can assume that the differentials of W have pairwise different orders at Q_1 (cf. [13, Remark 1.2]). Thus, at most a_1 of them have order smaller than a_1 at Q_1. Normalize the remaining differentials making their orders pairwise different at Q_2; at most a_2 among these have order smaller than a_2. By proceeding in this way, we conclude that there exist at least $i-\deg(D_i)$ differentials ω in W such that $\operatorname{div}(\omega) \geqslant D_i$, reaching a contradiction. The theorem is proved. □

5. Curves with maximal osculating divisors

Keep the assumptions of the previous section. We study pointed curves that intersect maximally their osculating spaces, in the sense that the bound given in Theorem 4.5 is attained.

Definition 5.1. The sequence of osculating divisors of C at P is called *maximal* if $\deg(E_{s_i}) = i - 1$ for $i = 1, \ldots, d$.

Below, we present examples of curves whose sequence of osculating divisors is maximal. Note that, in this case, it follows from Theorem 4.1 that each special gap is a translation, that is, the Weierstrass semigroup is of maximal type.

Example 5.2 (*Kummer's extensions*)**.** Assume $\mathrm{char}(K) = 0$. Let n, r be coprime integers such that $n \geqslant 3$ and $1 \leqslant r < n - r$. Let C be the smooth curve whose function field is $K(x, y)$, where

$$y^n = x^{n-r}(x - c_1)^r(x - c_2) \cdots (x - c_n).$$

Here, x, y are transcendental over K and $c_1, \ldots, c_n$ are constants pairwise different. Then P_∞ (the pole of x) and $P_0, P_1, \ldots, P_n$ (the zeros of $x, x - c_1, \ldots, x - c_n$, respectively) are exactly the places of $K(x, y)$ that are (fully) ramified over $K(x)$. By the Riemann–Hurwitz formula, the genus of C is $n(n - 1)/2$.

For each $j = 1, \ldots, n - 1$, set $k_j := [rj/n]$. Note that $nk_j \leqslant rj - 1$, as n and r are coprime. It follows that the $n(n - 1)/2$ differentials

$$\omega_{i,j} := \frac{x^i(x - c_1)^{k_j}}{y^j}\,\mathrm{d}x, \qquad \begin{array}{l} j = 1, \ldots, n - 1, \\ i = j - k_j - 1, \ldots, 2j - k_j - 2, \end{array}$$

are regular and have pairwise different orders at P_∞. So, they form a basis for the space of the regular differentials on C. The gaps of C at P_∞ are

$$\{un + v \mid u = 0, \ldots, n - 2,\ v = 1, \ldots, n - 1 - u\}$$

or more conveniently,

$$\begin{array}{l} 1, \ldots, (n - 1), \\ n + 1, \ldots, 2(n - 1), \\ 2n + 1, \ldots, 3(n - 1), \\ \vdots \\ (n - 1)^2. \end{array}$$

Thus, $\ell_g = (n - 1)^2$, $d = n - 2$ and $s_j = j(n - 1)$ for $j = 1, \ldots, n - 2$. Therefore, the Weierstrass semigroup of C at P_∞ is of maximal type. On the other hand, the sequence of osculating divisors depends on the value of r.

Assume $r = 1$. Then $k_j = 0$ for every j and so the sequence of osculating divisors of C at P_∞ is

$$0, P_0, 2P_0, \ldots, (d - 1)P_0, dP_0$$

and so it is maximal. The osculating hyperplane meets C outside P_∞ only at P_0.

Suppose $r > 1$. Then the sequence of osculating divisors at P_∞ is (note that $n > 2r$)

$$\dots,(n-2r-1)P_0+(r-1)P_1,(n-r-1)P_0+(r-1)P_1$$

and so it is not maximal in this case. From these examples we conclude that, in general, the gap sequence does not determine the behavior of osculating spaces (compare with Corollary 4.3).

Example 5.3 (*Hermitian curves*)**.** Consider the plane curve C given by

$$Y^q+Y=X^{q+1},$$

where $q>2$ is a power of $\mathrm{char}(K)>0$. As a smooth projective plane curve, C has degree $q+1$ and genus $q(q-1)/2$. Let P be a nonWeierstrass point of C. Then the gap sequence at P is $\{uq+v \mid u=0,\dots,q-2,\ v=1,\dots,q-1-u\}$ (see, e.g., [5]). So, $d=q-2$ and the special gaps are $s_i=i(q-1)$ for $i=1,\dots,q-2$. Note that this is a gap sequence of a semigroup of maximal type (compare with Example 5.2). By choosing a P-hermitian basis for the regular differentials on C (cf. [5, Section 1]), we see that the osculating hyperplane of C at P meets C outside P at just one more point and $\deg(E_{\tau_i})=i-1$ for $i=1,\dots,d+1$. Hence, the sequence of osculating divisors at P is maximal.

We write the intersection divisor of C^* and the osculating hyperplane as

$$E_{\ell_g}=d_1Q_1+\cdots+d_mQ_m$$

and so $\sum d_i=d$. If the sequence of osculating divisors is maximal, then there is a partition on the set of special gaps, namely

$$S_i:=\{s_k \mid \mathrm{ord}_{Q_i}(E_{s_k})=\mathrm{ord}_{Q_i}(E_{s_{k+1}})-1\} \quad (i=1,\dots,m),$$

where $s_{d+1}:=\ell_g$. Put $S_i=\{s_{i,1}<s_{i,2}<\cdots<s_{i,d_i}\}$. After an eventual reordination of the Q_i's, we may assume

$$s_{m,d_m}<s_{m-1,d_{m-1}}<\cdots<s_{1,d_1}=s_d. \tag{4}$$

Proposition 5.4. *Suppose the sequence of osculating divisors of C at P is maximal. Let $\{\omega_\ell \mid \ell\in L\}$ be a P-hermitian basis for the regular differentials on C. Then, for each $i=1,\dots,m$ and $j=1,\dots,d_i$, the following assertions hold*:

(i) $\mathrm{ord}_{Q_i}(\omega_{s_{i,j}})=j-1$.

(ii) $s_{i,j}+s_{i,d_i+1-j}=\ell_g$.

In addition, after an eventual change of the basis, the following also hold.

(iii) $\mathrm{ord}_{Q_i}(\omega_\ell)\geqslant d_i$ *for each gap* $\ell\notin S_i$.

(iv) $\mathrm{div}(\omega_\ell)\geqslant E_{\ell_g}$ *for each nonspecial gap* ℓ.

Proof. Let $D_j:=E_{\ell_g}-E_{s_{d+1-j}}$ for $j=1,\dots,d$. We have $\deg(D_j)=j$, since the sequence of osculating divisors is maximal. As we have seen in the proof of Theorem 4.5, we have $\mathrm{div}(\omega_{s_j}) \not\geqslant D_j$ for each j. By (4), we have $D_1 = Q_1$ and so $\mathrm{ord}_{Q_1}(\omega_{s_1}) = 0$. Moreover, we have $\mathrm{div}(\omega_\ell) \geqslant Q_1$ for each gap $\ell > s_1$, because otherwise we could normalize $\omega_{s_1} \mapsto \omega_{s_1} + c\omega_\ell$, for a suitable constant $c \in K$, in order to obtain $\mathrm{ord}_{Q_1}(\omega_{s_1}) > 0$. So, it follows from (3) that $s_{1,1} = s_1$. Similarly, we have $\mathrm{div}(\omega_{s_2}) \not\geqslant D_2$ and $\mathrm{div}(\omega_\ell) \geqslant D_2$ for each gap $\ell > s_2$. By (3) and (4), the gap s_2 is equal to $s_{1,2}$ or $s_{2,1}$, according to $D_2 - D_1$ is Q_1 or Q_2. By proceeding in this way, we obtain successively for $j = 1, \ldots, d$,

$$D_{j-1} \leqslant \mathrm{div}(\omega_{s_j}), \quad \mathrm{div}(\omega_{s_j}) \not\geqslant D_j \text{ and } \mathrm{div}(\omega_\ell) \geqslant D_j \text{ for each gap } \ell > s_j,$$

and conclude that (i) and (ii) hold.

Let us prove (iii). Let ℓ be a gap and suppose $\ell \notin S_i$. We have proved that $\mathrm{ord}_{Q_i}(\omega_\ell) > j-1$ whenever $\ell > s_{i,j}$. So, by normalizing $\omega_\ell \mapsto \omega_\ell + \sum_{s>\ell} c_s\omega_s$, where s runs through S_i and the c_s's are suitable constants, we get $\mathrm{ord}_{Q_i}(\omega_\ell) \geqslant d_i$, as required. Since (iv) is a particular case of (iii), the proposition is proved. □

Below, we present a monomial basis for the space of n-fold regular differentials. Such bases were given in [17, Section 2] for symmetric semigroups, in [11, Theorem 2.3] for the quasi-symmetric case and in [13, Theorem 2.1] for semigroups whose type is two. To do that, we make local considerations not only at P, but also at the other points of the intersection between the curve and the osculating hyperplane.

Suppose the set of gaps of C at P is not $\{1, \ldots, g\}$. Given $r \in \{2, \ldots, \ell_g - 1\}$, we consider all pairs of gaps (a, b) such that

$$r = a + b \text{ and } a \text{ or } b \text{ is not a translation.}$$

By Theorem 2.2, there is at least one such pair. Let (a_r, b_r) be the pair with the smallest a. For $r = \ell_g$, set $a_{\ell_g} = s_{1,1}$ and $b_{\ell_g} = s_{1,d_1}$. Note that, as we have seen in the proof of Proposition 5.4, $s_{1,1} = s_1$.

Theorem 5.5. *Preserve the above notation. Suppose the sequence of osculating divisors of C at P is maximal. Assume $L \neq \{1, \ldots, g\}$ and $d \geqslant 2$. Let $\{\omega_\ell \mid \ell \in L\}$ be a P-hermitian basis for the regular differentials on C as given by Proposition* 5.4. *Then, for each $n \geqslant 2$, the $(2n-1)(g-1)$ monomial expressions*

$$\begin{aligned} &\omega_{\ell_1}^{k}\omega_{a_r}\omega_{b_r}\omega_{\ell_g}^{n-2-k}, \quad k = 0, \ldots, n-2, \ r = 2, \ldots, \ell_g, \\ &\omega_{\ell_j}\omega_{\ell_g}^{n-1}, \quad j = 1, \ldots, g, \\ &\omega_{s_{i,1}}^{k+1}\omega_{s_{i,j}}\omega_{\ell_g}^{n-2-k}, \quad i = 1, \ldots, m, \ j = 1, \ldots, d_i, \\ &\qquad\qquad k = 0, \ldots, n-2, \ (i,j,k) \neq (1, d_1, 0), \end{aligned}$$

form a basis for the n-fold regular differentials on C.

Proof. By the Riemann–Roch theorem, the space of the n-fold regular differentials on C has dimension $(2n-1)(g-1)$. Therefore, we need only to show that the above differentials are linearly independent.

The differentials in the two first rows have pairwise different orders at P and so are linearly independent. Let φ be a differential in the first row with $r \neq \ell_g$. The sequence of osculating divisors is maximal and so every special gap is a translation. Thus, we have that a_r or b_r is not a special gap. Hence, it follows from Proposition 5.4 (iv) that $\mathrm{div}(\varphi) \geqslant (n-1)E_{\ell_g}$. On the other hand, if φ is in the first row and $r = \ell_g$, then $\mathrm{div}(\varphi) \geqslant (n-1)E_{\ell_g} - Q_1$. Now, let $\psi_{i,j,k} := \omega_{s_{i,1}}^{k+1}\omega_{s_{i,j}}\omega_{\ell_g}^{n-2-k}$ be a differential in the last row. By Proposition 5.4, we have for each i, j, k,

$$\mathrm{ord}_{Q_i}(\psi_{i,j,k}) = (j-1) + (n-2-k)d_i < (n-1)d_i$$

and

$$\mathrm{ord}_{Q_u}(\psi_{i,j,k}) \geqslant nd_u \quad \text{for } u \neq i.$$

So, the differentials $\{\psi_{i,j,k}\}$ are linearly independent. Since $\mathrm{ord}_{Q_1}(\psi_{1,j,k}) < (n-1)d_1 - 1$ for $j \neq d_1$ or $k \neq 0$, we conclude that all the differentials listed are linearly independent, as required. □

It follows from Theorem 5.5 that the map

$$K[W_{\ell_1}, \ldots, W_{\ell_g}]_n \to \Omega_C^n(0)$$

given by $W_{\ell_i} \mapsto \omega_{\ell_i}$ is surjective for each $n \geqslant 2$, which is known as *Noether's Theorem*. In particular, the canonical curve $C \subset \mathbb{P}^{g-1}$ is arithmetically Cohen–Macaulay. If C is smooth, this means that C is projectively normal.

Corollary 5.6. *Let i, j, k be positive integers such that $i \leqslant m$ and $j + k \leqslant d_i$. Then $s_{i,j+k} \leqslant s_{i,j} + s_{i,k}$. Moreover, if $s_{i,j} + s_{i,k}$ is a gap, then the equality holds.*

Proof. Let $\{\omega_\ell \mid \ell \in L\}$ be a P-hermitian basis as given by Proposition 5.4. The quadratic differential $\omega_{\ell_g - s_{i,j}}\omega_{s_{i,j+k}}$ has order $k - 1 + d_i$ at Q_i and order greater than $\ell_g - 2$ at P. By Theorem 5.5, it belongs to the space spanned by $\{\omega_\ell\omega_{\ell_g} \mid \ell \leqslant s_{i,k}\}$, since $\mathrm{ord}_{Q_i}(\omega_{\ell_g}) = d_i$ and $\mathrm{ord}_{Q_i}(\omega_\ell) > k-1$ for each $\ell > s_{i,k}$. So, $\ell_g - s_{i,j} + s_{i,j+k} \leqslant s_{i,k} + \ell_g$ and the result follows.

Assume now that $s_{i,j} + s_{i,k}$ is a gap. By Theorem 5.5 we can write

$$\omega_{\ell_g - s_{i,j}}\omega_{s_{i,j}+s_{i,k}} = \sum_{\ell \geqslant s_{i,k}} a_\ell \omega_\ell \omega_{\ell_g} \quad (a_\ell \neq 0).$$

Since the right-hand side has order $k - 1 + d_i$ at Q_i, we have $\mathrm{ord}_{Q_i}(\omega_{s_{i,j}+s_{i,k}}) = k + j - 1$, that is, $s_{i,j} + s_{i,k} = s_{i,j+k}$. □

Remark 5.7. The semigroup $\mathbb{N} \setminus \{1,2,3,5,7\}$, or more generally the semigroups

$$\mathbb{N} \setminus \{1, 2, \ldots, 2k-1, 3k-1, 4k-1\} \quad \text{for } k \geqslant 2,$$

are examples of semigroups of maximal type that do not satisfy the conditions of Corollary 5.6. So, they cannot be realized as a Weierstrass semigroup of a pointed curve whose sequence of osculating divisors is maximal.

6. Certain Gorenstein curves

As we have seen in Remark 5.7, there are semigroups of maximal type that cannot be realized by pointed curves whose sequence of osculating divisors is maximal. However, Theorem 3.1 and Corollary 5.6 suggest how to realize a certain family of them (see Definition 6.2 below) by using reducible canonical curves. On the other hand, some semigroups of this family cannot be realized as Weierstrass semigroups of smooth curves, as we shall see in Proposition 6.5.

Let $N=\{0,n_1,n_2,\ldots\}$ be a nonsymmetric semigroup. Let $L=\{\ell_1,\ell_2,\ldots,\ell_g\}$ be its set of gaps, where $\ell_g=2g-1-d$. Let Γ be the set of translations of N and $t_1,t_2,\ldots,t_\tau$ be the translation gaps. We assume that N is of maximal type, that is, $\tau=d+1$ and so every special gap is a translation.

We associate to the semigroup N the *projective monomial curve*

$$X:=\{(a^{\ell_1-1}b^{\ell_g-\ell_1}:a^{\ell_2-1}b^{\ell_g-\ell_2}:\cdots:a^{\ell_g-1}b^{\ell_g-\ell_g})\in\mathbb{P}^{g-1}\,|\,(a:b)\in\mathbb{P}^1\}.$$

Thus, $X\subset\mathbb{P}^{g-1}$ is a nondegenerate, irreducible rational curve of degree $2g-2-d$. Its function field is generated by the function t given by

$$(a^{\ell_1-1}b^{\ell_g-\ell_1}:a^{\ell_2-1}b^{\ell_g-\ell_2}:\cdots:a^{\ell_g-1}b^{\ell_g-\ell_g})\mapsto\frac{a}{b}.$$

This function is a local parameter of X at the nonsingular point $(1:0:\cdots:0)$. The contact orders of X and hyperplanes "$x_{\ell_i}=0$" at this point are ℓ_i-1 for $i=1,\ldots,g$. The curve X has just one singular point, namely the unibranched point $(0:\cdots:0:1)$. We use the local parameter $x:=1/t$ to study the local ring of this singularity.

Proposition 6.1. *The completion of the local ring of the curve X at its singular point is*

$$\left\{\sum_{i=0}^{\infty}d_ix^i\in K[[x]]\,|\,d_\ell=0,\ \ell\in L\setminus\{t_1,t_2,\ldots,t_\tau\}\right\}.$$

This ring is Gorenstein if and only if $n_1=\tau+1$.

Proof. Let S be the set of special gaps of N. The completion of the local ring is

$$K[[x^{\ell_g-\ell_1},x^{\ell_g-\ell_2},\ldots,x^{\ell_g-\ell_{g-1}}]]$$

and so its semigroup of values is generated by

$$\{n\in N\,|\,n<\ell_g\}\cup S=\{n\in N\,|\,n<\ell_g\}\cup\{t_1,t_2,\ldots,t_\tau\}$$

since N is of maximal type and nonsymmetric. So, by Proposition 2.1 (iii), the semigroup of values is Γ and the completion is as asserted.

The local ring of an irreducible branch is Gorenstein if and only if its semigroup of values is symmetric (cf. [9]). The semigroup Γ has genus $g-\tau$ and its last gap is ℓ_g-n_1 and so it is symmetric if and only if $2(g-\tau)-1=2g-\tau-n_1$, that is, $n_1=\tau+1$. □

By Proposition 6.1, the singularity degree of X at its singular point is $g-\tau$. Since X is rational and does not have other singularities, its arithmetic genus is $g-\tau$ (cf. [7, Theorem 2]). So, a basis for regular differentials on X is given by $\{t^{\ell-1}\,\mathrm{d}t \mid \ell \in L\setminus\{t_1,t_2,\dots,t_\tau\}\}$.

Definition 6.2. Let $d_1,\dots,d_m$ be positive integers such that $\sum d_i = d$. A semigroup of maximal type is called $(d_1,\dots,d_m)$-*symmetric* if there exists a partition $\bigcup_{i=1}^m S_i$ of the set of special gaps, say $S_i=\{s_{i,1},s_{i,2},\dots,s_{i,d_i}\}$, such that $s_{i,j}=js_{i,1}$ for $j=1,\dots,d_i$ and $\ell_g=(d_i+1)s_{i,1}$ for each i.

From now on we assume that N is a $(d_1,\dots,d_m)$-symmetric semigroup whose gap sequence is not $1,\dots,g$. For convenience, set $s_{i,d_i+1}{:=}\ell_g$ for each i.

For $i=1,\dots,m$, consider the rational normal curve

$$Y_i{:=}\left\{(c_{\ell_1}:\cdots:c_{\ell_g})\in\mathbb{P}^{g-1}\;\middle|\;\begin{array}{ll} c_\ell=0 & \text{if } \ell\notin S_i\cup\{\ell_g\}\\ c_{s_{i,j}}=a^{j-1}b^{d_i+1-j} & \text{for } j=1,\dots,d_i+1\end{array}\right\},$$

where $(a:b)\in\mathbb{P}^1$. As a generator of $K(Y_i)$ we take the function y_i defined by

$$(c_{\ell_1}:\cdots:c_{\ell_g})\mapsto\frac{c_{s_{i,1}}}{c_{s_{i,2}}}.$$

This is a local parameter of Y_i at the point $(0:\cdots:0:1)$. Let $C\subset\mathbb{P}^{g-1}$ be the reduced curve given by the union of curves $X,Y_1,\dots,Y_m$. Since these curves meet only at the point $(0:\cdots:0:1)$, this is unique singular point of C. The algebra of rational functions on C is $K(C)=K(X)\times K(Y_1)\times\cdots\times K(Y_m)=K(x)\times K(y_1)\times\cdots\times K(y_m)$.

Proposition 6.3. *Let $A{:=}K[[x]]\times K[[y_1]]\times\cdots\times K[[y_m]]$. The completion of the local ring of C at its unique singular point is*

$$\left\{\left(\sum_{i=0}^\infty c_ix^i,\sum_{j=0}^\infty e_{1,j}y_1^j,\dots,\sum_{j=0}^\infty e_{m,j}y_m^j\right)\in A\;\middle|\;\begin{array}{l} c_\ell=0,\ \ell\in L\setminus\{t_1,t_2,\dots,t_\tau\}\\ c_0=e_{1,0}=\cdots=e_{m,0}\\ c_{s_{i,j}}=e_{i,j},\ j=1,\dots,d_i\\ c_{\ell_g}=e_{1,d_1+1}+\cdots+e_{m,d_m+1}\end{array}\right\}.$$

This is a local Gorenstein ring whose singularity degree is $g+m$.

Proof. Let $R{:=}(0:\cdots:0:1)$. The image of completion ring

$$\hat{\mathcal{O}}_{C,R}\hookrightarrow\hat{\mathcal{O}}_{X,R}\times\hat{\mathcal{O}}_{Y_1,R}\times\cdots\times\hat{\mathcal{O}}_{Y_m,R}$$

consists of elements $(f_0,f_1,\dots,f_m)\in A$ where

$$f_0=f(u(W_{\ell_1}),\dots,u(W_{\ell_{g-1}})),$$
$$f_i=f(u_i(W_{\ell_1}),\dots,u_i(W_{\ell_{g-1}}))\quad\text{for } i=1,\dots,m,$$

for some $f\in K[[W_{\ell_1},\dots,W_{\ell_{g-1}}]]$ and

$$u(W_{\ell_j}){:=}x^{\ell_g-\ell_j},\qquad u_i(W_{\ell_j}){:=}\begin{cases}0 & \text{if } \ell_j\notin S_i\cup\{\ell_g\}\\ y_i^{d_i+1-k} & \text{if } \ell_j=s_{i,k},\ k=1,\dots,d_i\end{cases}$$

and so the asserted relations between the coefficients hold. In particular, the singularity degree of the local ring of C at R is at most

$$g - \tau + m + \sum_{i=1}^{m} d_i + 1 = g + m.$$

On the other hand, the codimension of its conductor in its integral closure A is at least $2g - d + \sum_{i=1}^{m}(d_i + 2) = 2g + 2m$. Since this codimension is at most twice the singularity degree (e.g. [16, p. 80]), it follows that the equality holds. Therefore, there are no other relations between the coefficients and $\hat{\mathcal{O}}_{C,R}$ is Gorenstein, as required. □

Corollary 6.4. *The curve $C \subset \mathbb{P}^{g-1}$ is a canonical curve of arithmetic genus g whose gaps at the point $(1:0:\cdots:0)$ are the integers $\ell_1, \ell_2, \ldots, \ell_g$. Moreover, its sequence of osculating divisors at this point is maximal.*

Proof. The normalization $\tilde{C}$ of C is the disjoint union of the rational nonsingular curves $\tilde{X}, Y_1, \ldots, Y_m$ and so $\tilde{C}$ has arithmetic genus $-m$. We conclude that the curve C has arithmetic genus g (cf. [7, Theorem 2]). By Proposition 6.3, the g differentials

$$\begin{aligned}
&(t^{\ell-1}\,\mathrm{d}t, 0, \ldots, 0), \quad \ell \in L \setminus \{t_1, t_2, \ldots, t_\tau\},\\
&(t^{s_{i,j}-1}\,\mathrm{d}t, \mathbf{0}, y_i^{-j-1}\,\mathrm{d}y_i, \mathbf{0}), \quad i = 1, \ldots, m, \; j = 1, \ldots, d_i,\\
&(t^{\ell_g - 1}\,\mathrm{d}t, y_1^{-d_1-2}\,\mathrm{d}y_1, \ldots, y_m^{-d_m-2}\,\mathrm{d}y_m)
\end{aligned}$$

form a basis for the regular differentials and so C is a canonical curve. From the differentials on the second line we conclude that the sequence of osculating divisors is maximal. □

As is well known, a necessary condition to a numerical semigroup is realizable as a Weierstrass semigroup of some smooth pointed curve (the "Buchweitz's criterion") in that its sets of sums of gaps must satisfy $\#L_n \leqslant (2n-1)(g-1)$ for $n \geqslant 2$. See [8] for examples of semigroups that do not satisfy this condition. By Proposition 2.5 (iii), these inequalities are satisfied for semigroups of maximal type. However, as has been shown in symmetric and quasi-symmetric cases (cf. [18, Scholium 3.5; 11, Theorem 5.1], respectively) this condition is not always sufficient. This is the case for some other semigroups of maximal type.

Proposition 6.5. *Let p be a prime number. Let H be a nonrealizable semigroup of genus h. Let g be an integer such that $g > (2p-1)(ph + p - 1)$ and $p - 1$ divides $2g$ but p does not divide s, where $s := (2g/(p-1)) - 1$. Let*

$$\begin{aligned}
N := pH \cup (s + p\mathbb{N}^+) \cup (2s + p\mathbb{N}^+) \cup \cdots \cup ((p-2)s + p\mathbb{N}^+)\\
\cup \{(p-1)s - pj \mid j \in \mathbb{Z} \setminus H\},
\end{aligned}$$

where $\mathbb{N}^+$ is the set of the positive integers. Then N is a $(p-2)$-symmetric semigroup of genus g which is not realizable as a Weierstrass semigroup of a smooth curve.

Proof. Since the prime p does not divide s, the classes of $p, s, 2s, \ldots, (p-1)s$ modulo p are pairwise different. Thus, the above union is disjoint, and therefore N is a semigroup. Its last gap is $(p-1)s = 2g-1-(p-2)$ and its special gaps are $s, 2s, \ldots, (p-2)s$. So, N is a $(p-2)$-symmetric semigroup of genus g.

Now we argue as in the quasi-symmetric case (see [11, Theorem 5.1] for details): suppose C is a smooth curve whose Weierstrass semigroup at P is N. Let f be the morphism associated to the base-point-free linear system $|n_{h+2}P|$. It follows from Castelnuovo's bound [1, p. 116] that f is not birational. So, f is a morphism of degree p (since p is prime) that carries C onto a smooth curve having at $f(P)$ the semigroup H (for the last step one uses Puiseux's Theorem), reaching a contradiction. □

As Prof. Torres pointed to me, the semigroups given in Proposition 6.5 appear implicitly in the family given by him in [19, Corollary 4.2.1].

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