On Fibrations by Nonsmooth Curves

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Zariski (1944) discovered that Bertini's theorem on variable singular points may fail in positive characteristic.

Counter-Example in Positive Characteristic

Let us consider k be an algebraically closed field of characteristic $p > 0$

$$S := \{((x, y), t) \in \mathbb{A}^2(k) \times \mathbb{A}^1(k) \mid y^2 + x^p - t = 0\}$$

and

$$\eta : S \to \mathbb{A}^1(k)$$

induced by the restriction to S of second projection $\mathbb{A}^2(k) \times \mathbb{A}^1(k) \to \mathbb{A}^1(k)$.

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$$\mathrm{Sing}(\eta^{-1}(t)) = \{((t^{1/p}, 0), t)\}$$

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1. Fibrations by cusps arose in the extension of Enriques' classification of surfaces to positive characteristic, due to Bombieri and Mumford (1976), in order to characterize quasi-hyperelliptic surfaces;
2. It enables us to find other geometrical constructions that never occur in characteristic zero. For example, in (S., 2011) we can find a linear system of nonclassical curves in $\mathbb{P}^2$, given by

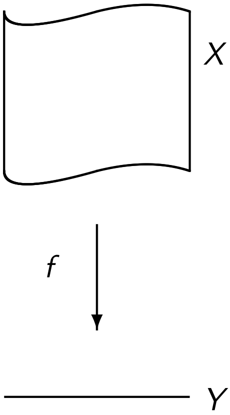
$$\mathbb{P}^2(k) \dashrightarrow \mathbb{P}^1(k)$$

defined by the assignment $(x:y:z) \mapsto (z^4 : y^3z - x^4)$, where k is an algebraically closed field of characteristic 3.

Links

Let us consider $f : X \to Y$ a fibration by curves, between varieties (integral) over an algebraically closed field k, that is:

- f is a dominant and proper morphism;
- Almost all fibers of f are integral curves;
- X is smooth, up to a base restriction to a dense open subset of Y.



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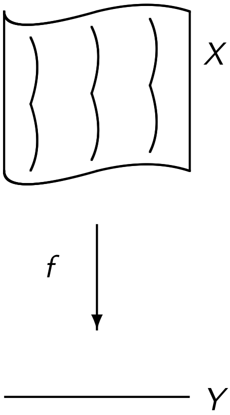

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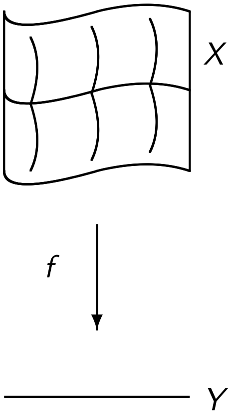

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If η is the generic point of Y, then we have the following bijection

$$\left\{ \begin{array}{c} \text{horizontal prime} \\ \text{divisors of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{closed points of} \\ X_\eta = X \times_Y \operatorname{Spec} k(Y) \end{array} \right\}$$

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Question: What property characterizes the closed points of X_η corresponding to the horizontal prime divisors contained in the nonsmooth locus of f?

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To answer this question we remind the too different concepts of a simple point P on a variety V (over a field K), due to Zariski (1947):

1. Regular in the sense of having a regular local ring;
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$$\left\{ \begin{array}{c} \text{horizontal prime} \\ \text{divisors contained in} \\ \text{the singular locus of} \\ f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{closed and} \\ \text{nonsmooth points} \\ \text{of } X_\eta \end{array} \right\}$$

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f is a fibration by non smooth curves

$$\Longleftrightarrow$$

X_η is regular but nonsmooth,
geometricaly integral and complete
curve over $k(Y)$

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In this case, the function field of one variable $K(C)|K$ is classically called nonconservative.

The Relative Frobenius Effect

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Now we will just consider schemes over the finite field $\mathbb{F}_p$, where $p > 0$ is a fixed prime number. Given a scheme S we have the absolute Frobenius morphism of S

$$F_S : S \to S$$

induced by the following ring homomorphism

$$\begin{array}{ccc} \mathcal{O}_S & \to & \mathcal{O}_S \\ a & \mapsto & a^p \end{array}$$

The Relative Frobenius Effect

In addition, if we consider a scheme X over S, we may consider other S-scheme $X^{(p)} = X \times_S S$ obtained by pulling back of π via F_S.

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F_S} & S \end{array}$$

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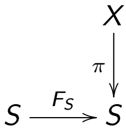
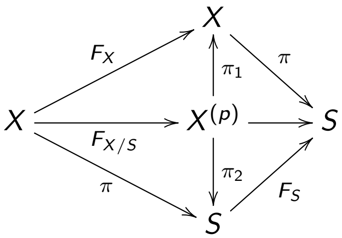


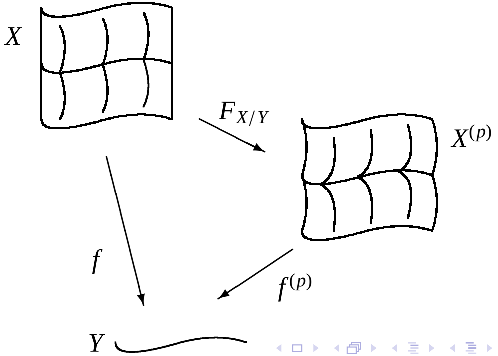
The relative Frobenius morphism is the unique morphism $F_{X/S}$ that commutes the following diagram.



The Relative Frobenius Effect

Proposition (S., 2011)

Let $f : X \to Y$ be a proper dominant morphism between algebraic varieties with a geometrically integral curve as generic fiber. Then, the images of the horizontal prime divisors, contained in the nonsmooth locus of f, under $F_{X/Y}$, are precisely the horizontal prime divisors contained in the nonsmooth locus of $X^{(p)}$, as a variety over k.



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Corollary (S.)

The singularities appearing on the general fiber of a fibration by nonsmooth curves are always unibranch.

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5. S. (2011) started to classify the case $g = 3$ and $p = 3$.

The geometric singularity degree of $P \in C$ is defined by

$$\dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P,C}}}{\overline{K}\mathcal{O}_{P,C}},$$

where $\widetilde{\overline{K}\mathcal{O}_{P,C}}$ is the integral closure of $\overline{K}\mathcal{O}_{P,C}$ in the field $K(C)\overline{K} \simeq \overline{K}(C \otimes_K \overline{K})$.

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It can be proved that the geometric singularity degree is a multiple of $(p-1)/2$ and

$$g - \overline{g} = \sum_{P \in C} \dim_{\overline{K}} \frac{\widetilde{\overline{K}\mathcal{O}_{P,C}}}{\overline{K}\mathcal{O}_{P,C}},$$

where $\overline{g}$ is the geometric genus of $C \otimes_K \overline{K}$.

Regular but Nonsmooth Curves

For the case $p = 3$ and $g = 3$, we have the following possibilities for the geometric genus $\overline{g}$ of $C \otimes_K \overline{K}$, the number of nonsmooth points of C and their geometric singularity degrees.

$\overline{g}$	Number of Nonsmooth Points	Possible Geometric Singularity Degrees
o	1	3
	2	1 and 2
	3	1
1	1	2
	2	1
2	1	1

Regular but Nonsmooth Curves

Theorem (S.)

Let C be a regular and complete algebraic curve over a field K of characteristic three. Then C is geometrically integral of genus three and admits three nonsmooth points if and only if it is isomorphic to a plane projective quartic curve over K defined by the homogeneous polynomial

$$ZY^3 - (a+b+c)X^4 - (a-b)X^3Z - (a+b+c)X^2Z^2 + cZ^4$$

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where $a, b, c \in K \setminus K^3$ and $a+b+c \neq 0$. Moreover, the nonsmooth points of such a curve correspond to the points lying under the points $(0 : c^{1/3} : 1)$, $(1 : b^{1/3} : 1)$ and $(-1 : a^{1/3} : 1)$ of the projective plane $\mathbb{P}^2(\overline{K})$.

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Corollary

The extended curve $C \otimes_K \overline{K}$ is nonclassical and admits just one smooth Weierstrass point.

Universal Fibration of Nonsmooth Curves

In order to obtain fibrations by nonsmooth curves and investigate their geometric properties, we construct a universal fibration in the sense that the data about all fibrations by nonsmooth plane quartics, whose generic fiber are previously fixed, are condensed in it.

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We consider the rational 4-fold T in $\mathbb{P}^2 \times \mathbb{A}^3$ given by the polynomial $ZY^3 - (a+b+c)X^4 - (a-b)X^3Z - (a+b+c)X^2Z^2 + cZ^4$ and $\pi : T \to \mathbb{A}^3$ induced by the second projection.

Theorem (S.)

Each fibration by nonsmooth genus three curves with three nonsmooth points, is up to birational equivalence obtained by a base extension, under a dominant morphism, either from $\pi : T \to \mathbb{A}^2$ or from a fibration obtained by restricting the base of π to an irreducible curve on $\mathbb{A}^3$ or to an irreducible surface on $\mathbb{A}^3$.

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Corollary

Almost all fibers of a fibration by nonsmooth plane projective quartic curves, with three nonsmooth points, are nonclassical curves and admit a unique Weierstrass point.

Interesting Problems

1. To describe the minimal proper regular model of fibrations by nonsmooth curves over the projective line and determine the structure of their fibers, in analogy to the Kodaira–Néron classification of special fibers of minimal fibrations by elliptic curves.

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2. What kind of surfeces can be fibered by nonsmooth curves?