Hypersurface singularity in arbitrary characteristic

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24^{th} Brazilian Algebra Meeting

02 August, 2016

Joint work with
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Therefore, $p = 0$ and $\tau(f) < \infty \Longrightarrow \mu(f) < \infty$.

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Theorem. Let $f \in k[X_1, \cdots, X_n]$ admitting an isolated singularity at the origin of $\mathbb{A}^n_k$. The fibration $f\colon \mathbb{A}^n_k \to \mathbb{A}^1_k$ is a local smoothing at $0 \in f^{-1}(0)$ if and only if $\mu_0(f) := \dim_k \frac{\mathcal{O}_{\mathbb{A}^n_k,0}}{J(f)} = \mu(f) < \infty$.

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Then in the completion $\widehat{\mathcal{O}_{S,P}}$ of the local ring there exist local parameters x and y such that $D \sim \frac{\partial}{\partial y}$.

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Proposition. Let $f \in k[X, Y]$ vanishing and with isolated singularity at the origin $0 \in \mathbb{A}^2_k$, where k is an algebraically closed field of characteristic $p > 0$. Suppose that $\mu(f) = \infty$ and set
$h = \gcd(f_X, f_Y) \in \mathfrak{m} \subset \mathcal{R} = k[[X, Y]]$.

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Proposition. Let $f \in k[X, Y]$ vanishing and with isolated singularity at the origin $0 \in \mathbb{A}^2_k$, where k is an algebraically closed field of characteristic $p > 0$. Suppose that $\mu(f) = \infty$ and set $h = \gcd(f_X, f_Y) \in \mathfrak{m} \subset \mathcal{R} = k[[X, Y]]$. Then there exists an automorphism ϕ of $\mathcal{R}$ such that $\phi(f) \in k[[X, Y^p]]$ if and only if f_X/h or f_Y/h does not belong to $\mathfrak{m}$.

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$h = \text{gcd}(f_X, f_Y) = Y - X$ and $f_X/h = Y, f_Y/h = -X \in \mathfrak{m}$.

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We denote by $e_0(I)$ the Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal I of $\mathcal{R}$ and we put $e_0(I) = \infty$ if I is not $\mathfrak{m}$-primary.

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$\Longrightarrow f \notin \overline{J(f)}$.

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We call an equation f of $\mathcal{O}$ a good equation if $f \in \overline{J(f)}$.

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Good news: good equations are “generic” in the contact class.

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Theorem. Let $f \in \mathfrak{m}$ with $\tau(f) < \infty$ and

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Hence $f \in \mathfrak{m}T(f)$ and is μ-stable.

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Two plane branches f and h are called equisingular when

$$S(f) = S(h).$$

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Note that p divides one of the generators of $S(f)$.

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(García-Barroso and Ploski, 2015) The equivalence holds if

$$p > v_0 = \text{mult}(f).$$

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(Zariski, 1966): If f is a plane branch and char $k = 0$, then $c(f) = \tau(f) \iff$ up to change coordinates $f = Y^n - X^m$ with n and m relatively primes.

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THANK YOU!