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HYPERSURFACE SINGULARITIES IN ARBITRARY CHARACTERISTIC

Niterói
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JOÃO HÉLDER OLMEDO RODRIGUES

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Tese apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática e Estatística da Universidade Federal Fluminense, como requisito parcial à obtenção do título de Doutor em Matemática

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Coorientador: Rodrigo Salomão

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Ata dos trabalhos finais da Comissão Examinadora da Tese de Doutorado apresentada por João Hélder Olmedo Rodrigues

Aos treze dias do mês de novembro de dois mil e quinze, reuniram-se no auditório da Pós-Graduação em Matemática da Universidade Federal Fluminense, os membros da Comissão Examinadora constituída pelos professores Abramo Hefez, da Universidade Federal Fluminense; Rodrigo Salomão, da Universidade Federal Fluminense; Marcelo Escudeiro Hernandes, da Universidade Estadual de Maringá; Roberto Callejas Bedregal, da Universidade Federal da Paraíba e Karl-Otto Stöhr, do Instituto Nacional de Matemática Pura e Aplicada, sob a presidência do primeiro, para prova pública de defesa da tese intitulada **"Singularidades de Hipersuperfícies em Característica Arbitrária"**, apresentada pelo doutorando João Hélder Olmedo Rodrigues. A defesa da tese atende às exigências contidas no Regulamento Específico do Curso de Doutorado em Matemática da Universidade Federal Fluminense. A tese foi elaborada sob a orientação do professor Abramo Hefez e co-orientado pelo professor Rodrigo Salomão. O doutorando João Hélder Olmedo Rodrigues fez a exposição de seu trabalho durante 50 minutos, iniciando às 10h00 e concluindo às 10h50. A seguir, respondeu as questões formuladas pelos integrantes da Comissão Examinadora. Terminada a arguição, realizou-se a reunião da Comissão Examinadora, que apresentou parecer no sentido da aprovação do doutorando João Hélder Olmedo Rodrigues, considerando-se o trabalho apresentado e a forma com que se houve na apresentação da defesa do mesmo. Para constar, foi lavrada a presente ata, que vai assinada pela Secretária Administrativa da Coordenação de Pós-Graduação em Matemática, pelos membros da Banca Examinadora e pelo doutorando.

Niterói, 13 de novembro de 2015.

Abramo Hefez
Prof. Abramo Hefez

Prof. Rodrigo Salomão

Prof. Marcelo Escudeiro Hernandes

Prof. Roberto Callejas Bedregal

Karl Otto Stöhr
Prof. Karl-Otto Stöhr

João Hélder Olmedo Rodrigues
João Hélder Olmedo Rodrigues - Doutorando

Jacqueline Garcez Tavolaro
Jacqueline Garcez Tavolaro (Secretária)

À vida (e suas desconcertantes reviravoltas).

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DE RISO E ROSA

Tenho o que preciso pra acabar com o juízo
Tenho o que preciso
Um chão que se desmina a cada passo, piso
Tenho o que preciso
Deslizo meu riso, grito o meu aviso: Odeio o juízo!
A loucura tanto bate até que fura a pedra da convicção
A loucura tanto bate até que fura a tua dura razão.
E a razão é rasinha, amor, rasinha
É raiz morta
A razão é rasinha, amor, ô ô
De riso e rosa.

(Claos Mózi)

Resumo

O estudo de singularidades de espaços analíticos ou algébricos sobre o corpo dos números complexos é um tema tradicional que tem visto avanços impressionantes nas últimas décadas. Em contraste, uma teoria paralela sobre corpos algebricamente fechados é ainda pobre e existem muitas questões interessantes para serem respondidas. Nosso objetivo nessa tese é contribuir nessa direção focando em algumas questões sobre singularidades de hipersuperfícies e, mais particularmente, sobre singularidades de curvas planas.

Nosso ponto principal aqui será o estudo do número de Milnor de uma singularidade isolada de uma hipersuperfície, o qual é definido como a codimensão do ideal gerado pelas derivadas parciais de uma série de potências que representa localmente a hipersuperfície. Este é um invariante topológico importante sobre os números complexos, mas seu significado muda dramaticamente quando o corpo de base é arbitrário. Acontece que, se o corpo é de característica positiva, este número pode ser infinito e depender da equação local da hipersuperfície. Nessa tese estudaremos a variação do número de Milnor em termos de uma equação local dando condições necessárias e suficientes para sua invariância. Nós também relacionamos, para uma singularidade isolada, a finitude desse número com a suavidade de uma fibra genérica de uma deformação da hipersuperfície, relacionando isso com um resultado de Bertini. Finalmente nós especializamos ao caso de singularidades irredutíveis de curvas planas onde damos uma condição suficiente em termos de um invariante de equisingularidade para a validade de um resultado de Milnor, conhecido sobre os números complexos, que diz que o número de Milnor em um ponto coincide com o condutor da curva naquele ponto. Concluímos o trabalho com o estudo do módulo de diferenciais de Kähler de uma curva plana sobre corpos de característica positiva, evidenciando várias diferenças com o caso de característica zero.

Palavras-chave: Singularidades em característica positiva, Número de Milnor em característica positiva, Singularidades de hipersuperfícies algebróides, Fibrações por hipersuperfícies não lisas.

Abstract

The study of singularities of algebraic or analytic spaces over the field of complex numbers is a traditional subject that has seen impressive developments in several directions in the last decades. In contrast, the parallel theory over arbitrary algebraically closed fields is still poor and there are lots of interesting questions to be answered. Our aim in this thesis is to contribute in that direction by focusing on some questions about hypersurface singularities and, more particularly, about plane curve singularities.

Our main concern here will be the study of the Milnor number of an isolated hypersurface singularity which is defined as the codimension of the ideal generated by the partial derivatives of a power series that represents locally the hypersurface. This is an important topological invariant of the singularity over the complex numbers, but its meaning changes dramatically when the base field is arbitrary. It turns out that, if the ground field is of positive characteristic, this number may be infinite and depends upon the local equation of the hypersurface, not being an intrinsic invariant of the hypersurface. In this thesis we will study the variation of the Milnor number in terms of a local equation, giving necessary and sufficient conditions for its invariance. We also relate, for an isolated singularity, the finiteness of this number to the smoothness of the generic fiber of a deformation of the hypersurface, relating it to a Bertini type result. Finally, we specialize to the case of plane irreducible curves where we give a sufficient condition in terms of an equisingularity invariant for the validity of a result of Milnor, known to be true over the complex numbers, that asserts that the Milnor number at a point coincides with the conductor of the curve at that point. We conclude the work with the study of the module of Kähler differentials of plane curves over fields of positive characteristic, evidencing many differences from the characteristic zero case.

Keywords: Singularities in positive characteristic, Milnor number in positive characteristic, Singularities of algebroid hypersurfaces, Fibrations by non-smooth hypersurfaces

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CHAPTER 1

Introduction

The local study of singularities of algebraic varieties over an arbitrary algebraically closed field may be reduced to the study of *algebroid varieties*.

By an algebroid variety over a field k we mean $\operatorname{Spec}\mathcal{O}$ where $\mathcal{O}$ is a noetherian local complete ring with coefficient field k. To simplify notation we will also call $\mathcal{O}$ an algebroid variety. If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, then $\dim \mathfrak{m}/\mathfrak{m}^2$ is the embedding dimension of $\mathcal{O}$ and it is well known that $\dim \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim \mathcal{O}$. The algebroid variety will be said *smooth* if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}$, otherwise it will be said singular. Our main concern here will be the singular case.

Let us set $n = \dim \mathfrak{m}/\mathfrak{m}^2$. We denote by $\mathcal{R} = k[[X_1, \ldots, X_n]]$ the ring of formal power series in n indeterminates with coefficients in k and by $\mathcal{R}^*$ the group of its units. If we choose a minimal set of generators $\{x_1, \ldots, x_n\}$ of $\mathfrak{m}$, from the completeness of $\mathcal{O}$, we get a k-algebras epimorphism

$$\begin{array}{rccc} \varphi\colon & \mathcal{R} & \to & \mathcal{O}. \\ & X_i & \mapsto & x_i \end{array}$$

Any such choice gives us a closed embedding $\operatorname{Spec}\mathcal{O} \hookrightarrow (\mathbb{A}^n_k, 0)$, where $(\mathbb{A}^n_k, 0) = \operatorname{Spec}\mathcal{R}$. If $\ker\varphi = \mathfrak{A}$, then we have

$$\mathcal{O} \simeq \mathcal{R}\,/\,\mathfrak{A}.$$

Another choice of generators of $\mathfrak{m}$ will produce a new embedding and consequently an ideal $\mathfrak{B}$ such that

$$\mathcal{R}\,/\,\mathfrak{A} \simeq \mathcal{O} \simeq \mathcal{R}\,/\,\mathfrak{B}.$$

The ideals $\mathfrak{A}$ and $\mathfrak{B}$ are related by the existence of an automorphism Φ of $\mathcal{R}$ that transforms $\mathfrak{A}$ into $\mathfrak{B}$.

Now, if $\mathcal{O}$ is a singular hypersurface, that is if $\dim \mathcal{O} = n-1$, then the ideal $\mathfrak{A}$ is a principal ideal $\langle f \rangle$, uniquely determined modulo an automorphism Φ of $\mathcal{R}$. From now on we only consider algebroid singular hypersurfaces. When $n = 2$, the singular hypersurface is called a plane curve and in this case we write $\mathcal{R} = k[[X, Y]]$.

A generator f of the ideal $\mathfrak{A}$ of a hypersurface embedded in $(\mathbb{A}^n_k, 0)$ will be called an equation of $\mathcal{O}$. Two generators of the ideal $\mathfrak{A}$ are associated.

Conversely, given $f \in \mathcal{R}$ not zero nor invertible, we associate to it the hypersurface denoted by $\mathcal{O}_f = \mathcal{R}/\langle f \rangle$. So, two elements f and g in $\mathcal{R}$ will determine the same algebroid hypersurface if and only if there is an automorphism Φ and a unit $u \in \mathcal{R}^*$ such that $g = u\Phi(f)$. In such case we say that f and g are *contact equivalent*.

Given $f \in \mathcal{R}$, we define the *Tjurina ideal* in $\mathcal{R}$ as being the ideal

$$T(f) = \langle f, f_{X_1}, \ldots, f_{X_n} \rangle.$$

This ideal plays a fundamental role in our presentation. The *Tjurina Algebra* of f is the algebra $\mathcal{R}/T(f)$ and its dimension $\tau(f)$ as a k-vector space is the *Tjurina number* of f.

It is easy to check that this number is invariant by contact equivalence, so it defines an invariant $\tau(\mathcal{O})$ of $\mathcal{O}$.

Another ideal that plays an important role when $k = \mathbb{C}$ is the *Jacobian ideal* $J(f)$ of f which is the ideal generated by all the partial derivatives of f:

$$J(f) = \langle f_{X_1}, \ldots, f_{X_n} \rangle.$$

The *Milnor Algebra* of f is the algebra $\mathcal{R}/J(f)$ and its dimension $\mu(f)$ as a k-vector space is the *Milnor number* of f. It is immediate to verify that $\mu(f) = \mu(\Phi(f))$ for any automorphism Φ of $\mathcal{R}$.

When $\mathcal{R} = \mathbb{C}\{X_1, \ldots, X_n\}$ is the convergent power series ring, Milnor proves by topological methods that $\mu(uf) = \mu(f)$ for any $u \in \mathcal{R}^*$. This result is usually extended over characteristic zero fields by Lefschetz' principle. In arbitrary characteristic this does not hold as one can see from the example below.

Example 1.1.1. *Let* $\operatorname{char} k = p$ *and* $f = Y^p + X^{p+1} \in k[[X,Y]]$. *Then* $\mathcal{O}_f$ *is an irreducible plane curve such that* $\tau(f) = p^2$ *and*

$$\mu(f) = \infty, \qquad \text{but} \qquad \mu((1+Y)f) = p^2 \neq \mu(f).$$

This is a relevant issue in our investigation since some important problems are connected to it. For instance, we will characterize in Section 2.1 the reduced $f \in k[[X,Y]]$ for which there exists an automorphism Φ of $k[[X,Y]]$ such that $\Phi(f) \in k[[X^p, Y]]$, where $p = \operatorname{char}(k)$. This is a sufficient (but not necessary) condition for the infiniteness of $\mu(f)$. In Section 2.3, we characterize the f for which $\mu(f) = \mu(uf)$ for all $u \in \mathcal{R}^*$ and will study in general the variation of $\mu(uf)$ when f is fixed and u varies in $\mathcal{R}^*$.

From the inclusion $J(f) \subset T(f)$, it is clear that

$$\tau(f) \leqslant \mu(f).$$

So, one always has

$$\mu(f) < \infty \ \Rightarrow \tau(f) < \infty.$$

In characteristic zero, one also has the converse of the above implication. This will be proved algebraically in Section 2.2. In positive characteristic, the converse may fail, as one can see from Example 1.1.1 above.

Motivated by the above discussion and by the fact that the ideal of a singularity of a hypersurface is the Tjurina ideal, the natural definition for isolated singularity is the following:

Definition 1.1.2. *A hypersurface* $\mathcal{O}_f$ *has an isolated singularity at the origin if* $f \in \mathfrak{m}^2$ *and* $\tau(f) < \infty$.

Notice that this is a well posed definition, since $\tau(f) = \tau(g)$ when f and g are contact equivalent. So, in characteristic zero, to say that $\mathcal{O}_f$ has an isolated singularity is equivalent to say that $\mu(f) < \infty$, but not in arbitrary characteristic.

There is an easy criterion in arbitrary characteristic (cf. [B], Proposition 1.2.11) for a plane curve $\mathcal{O}_f$ to have an isolated singularity: $\mathcal{O}_f$ has an isolated singularity if and only if f is reduced. In contrast, the fact that f is reduced is not sufficient to guarantee that $\mu(f) < \infty$ as shows Example 1.1.1. Also, the vanishing of one of the partial derivatives of f implies $\mu(f) = \infty$, but this is not a necessary condition, even in the case of plane curves, as the following example shows.

Example 1.1.3. *Let char$k = 3$ and $f = X^2Y + Y^2X \in k[[X,Y]]$. We have that $f = XY(X+Y)$ is reduced, but f_X and f_Y are both nonzero and have the common factor $Y - X$, implying that $\mu(f) = \infty$.*

In the next section we will give a criterion for the finiteness of $\mu(f)$ which will shed light on why in characteristic zero $\tau(f) < \infty$ implies $\mu(f) < \infty$.

Recall that in the complex case the Milnor number of f was introduced in [Mi] as the rank of the middle cohomology group of the fiber of the local smoothing $f = s$. In this setting $\mu(f)$ is referred to as the number of vanishing cycles associated to f. When we switch to a field of positive characteristic p, the fibration $f = s$ may not be a local smoothing anymore, that is, it may be a counter example to Bertini's theorem on the variation of singular points in linear systems, true in characteristic zero. In Section 2.1, we characterize this phenomenon, that may only occur in positive characteristic, in terms of the infiniteness of the Milnor number.

Finally, in the last two sections, we study plane branches singularities over arbitrary algebraically closed fields. In characteristic zero, the Milnor number $\mu(f)$ coincides with the conductor $c(f)$ of the semigroup of values of a branch (f). In arbitrary characteristic, Deligne proves in [De] (see also [MH-W]) the inequality $\mu(f) \geqslant c(f)$, where the difference $\mu(f) - c(f)$ measures the existence of wild vanishing cycles. We prove that Milnor's number and the conductor of a branch (f) coincide when the characteristic does not divide any of the minimal generators of the semigroup of values of f, which we call *tame semigroup*. Our proof was inspired by a result of P. Jaworski in the work [Ja2], which we simplified and extended to arbitrary characteristic, under the appropriate assumptions. We would like to point out that in the process of writing the final version of this thesis, E. García Barroso and A. Ploski posted the preprint [GB-P], showing by other methods our last result in the chapter 3 (with the converse), but in the much easier particular case when p is greater than the multiplicity of f, and also observed that their proof fails when p is less or equal than the multiplicity of f. We should also mention that H.D. Nguyen in [Ng] has shown, in the irreducible case, the also much weaker result, namely, that if $p > c(f) + \mathrm{mult}(f) - 1$, then $\mu(f) = c(f)$. Notice that once fixed the ground field k of positive characteristic p, both results in [GB-P] and [Ng] cover only finitely many values of the multiplicity $\mathrm{mult}(f)$, while our result is in full generality.

The last chapter of this thesis is dedicated to the study of the module of Kähler differentials of singularities of curves, where we show some similarities among the characteristic zero case and the branches with tame semigroups, for instance that the nonzero values of functions are values of differentials, and some differences, as the non validity of the result of Zariski that classifies the branches that have no differentials whose values are not values of functions.

CHAPTER 2

Milnor number

2.1 The finiteness of Milnor Number

In this section we give a necessary and sufficient condition on an equation f of a hypersurface $\mathcal{O}$ to have $\mu(f) < \infty$. We also relate the finiteness of $\mu(f)$ with the condition that the fibration $f = s$ is a smoothing for the singularity $\mathcal{O}$, establishing necessary and sufficient conditions for the validity of one of Bertini's theorem in positive characteristic for this fibration.

To discuss the finiteness of $\mu(f)$ we must impose that $\tau(f) < \infty$, or equivalently that $\mathcal{O}_f$ has an isolated singularity, because otherwise $\mu(f) = \infty$. So, assume that $\tau(f) < \infty$ and look to the natural exact sequence of k-vector spaces

$$0 \to T(f)/J(f) \to \mathcal{R}/J(f) \to \mathcal{R}/T(f) \to 0.$$

Intuitively, if the element $f \in T(f) = \langle f \rangle + J(f)$ is *not too far* from the ideal $J(f)$ then

$$\mu(f) = \tau(f) + \dim_k T(f)/J(f)$$

must be finite. This can be made precise and we arrive at the following result which gives a criterion for the finiteness of $\mu(f)$.

Proposition 2.1.1. *Let $f \in \mathfrak{m}$ and suppose that $\tau(f) < \infty$. Then*

$$\mu(f) < \infty \;\Leftrightarrow\; f \in \sqrt{J(f)}.$$

Proof: Notice that when $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ both conditions hold trivially. So, we will be only concerned with the case $f \in \mathfrak{m}^2$.

Suppose that $\mu(f) < \infty$, then $J(f)$ is $\mathfrak{m}$-primary, hence $f \in \mathfrak{m} = \sqrt{J(f)}$.

Conversely, suppose that $f \in \sqrt{J(f)}$. Since $\tau(f) < \infty$, we have that $\sqrt{T(f)} = \mathfrak{m}$. Now,

$$T(f) = \langle f \rangle + J(f) \subset \sqrt{J(f)} \subset \mathfrak{m},$$

since $f \in \mathfrak{m}^2$. Taking radicals we get

$$\mathfrak{m} = \sqrt{T(f)} \subset \sqrt{J(f)} \subset \mathfrak{m}.$$

So, $\sqrt{J(f)} = \mathfrak{m}$, which in turn implies that $\mu(f) < \infty$. ■

Remark 2.1.2. *At this point, accordingly to the announced equivalence between the conditions $\tau(f) < \infty$ and $\mu(f) < \infty$ in characteristic zero we deduce that in this situation one always has $f \in \sqrt{J(f)}$. We will postpone the proof of this fact to the next section.*

The following result will be needed in Chapter 4 of the thesis.

Lemma 2.1.3. *([B], Lemma 1.2.13) Let $f \in \mathfrak{m} \subset \mathcal{R}$ be such that $J(f) \subset \langle f \rangle$. Then either $f = 0$, if $\operatorname{char} k = 0$, or $f = ug^p$ for some unit u of $\mathcal{R}$ and $g \in \mathcal{R}$, if $\operatorname{char} k = p > 0$.*

We now discuss an interesting connection between the finiteness of $\mu(f)$ and Bertini's Theorem.

Let $f \in k[X_1, \ldots, X_n]$ and assume that f has an isolated singularity at the origin of $\mathbb{A}^n_k$. By this we mean that in the local ring $\mathcal{O}_{\mathbb{A}^n_k,0} = k[X_1, \ldots, X_n]_{\langle X_1,\ldots,X_n\rangle}$ the ideal $T_0(f) := T(f)\,\mathcal{O}_{\mathbb{A}^n_k,0}$ has finite codimension:

$$\tau_0(f) := \dim_k\; \mathcal{O}_{\mathbb{A}^n_k,0}/T_0(f) < \infty.$$

Accordingly, denote $\mu_0(f) := \dim_k\; \mathcal{O}_{\mathbb{A}^n_k,0}/J_0(f)$. Of course these numbers coincide with the preceding $\tau(f)$ and $\mu(f)$ since $\mathcal{R} \simeq \widehat{\mathcal{O}_{\mathbb{A}^n_k,0}}$.

We are going to study when the fibration by hypersurfaces induced by

$$f \colon \mathbb{A}^n_k \to \mathbb{A}^1_k$$

is a *local smoothing* of the singularity, that is, we will investigate under which conditions there are Zariski open neighbourhoods $U \subset \mathbb{A}^n_k$ and $V \subset \mathbb{A}^1_k$ of the origin $0 \in \mathbb{A}^n_k$ (respectively $0 \in \mathbb{A}^1_k$) such that

$$U \setminus f^{-1}(0) \to V \setminus 0$$

is smooth. Equivalently one can ask for the *non-existence* of horizontal subvarieties (varieties $C \subsetneq \mathbb{A}^n_k$ whose image dominates $\mathbb{A}^1_k$) through $0 \in \mathbb{A}^n_k$ consisting of singularities of the fibers. Notice that in characteristic zero this is always the case due to Bertini's theorem on the variation of singular points in linear systems. However, it is well known that this is not true over fields of positive characteristic.

Example 2.1.4. *We have already seen that $f = Y^p + X^{p+1}$ has an isolated singularity at the origin when $\operatorname{char} k = p > 0$. The fiber over each $s \in \mathbb{A}^1_k$ has $(0, s^{1/p})$ as a singularity.*

Example 2.1.5. *When $p = 3$ the polynomial $f = X^2Y + Y^2X$ has an isolated singularity at the origin because $\tau(f) = 4 < \infty$. Again, the fiber over each $s \in \mathbb{A}^1_k$ has $((-s)^{1/3}, (-s)^{1/3})$ as a singularity.*

In arbitrary characteristic we have the following characterization which was motivated by the proof of the Proposition 2.1.1.

Theorem 2.1.6. *Let f be a polynomial admitting an isolated singularity at the origin. The fibration $f\colon \mathbb{A}^n_k \to \mathbb{A}^1_k$ is a local smoothing if and only if $\mu_0(f) < \infty$.*

Proof: If $\mu_0(f) = \infty$, then the codimension of $J(f) = \langle f_{X_1}, \dots, f_{X_n} \rangle$ in $\mathcal{O}_{\mathbb{A}^n_k,0}$ is infinite. This implies that the sequence $f_{X_1}, \dots, f_{X_n}$ is not $\mathcal{O}_{\mathbb{A}^n_k,0}$-regular. In this case $Z(f_{X_1}, \dots, f_{X_n})$ contains a curve C trough the origin in $\mathbb{A}^n_k$. We clearly have that $C \cap Z(f - s)$ is made of singular points of the fiber $f = s$. Hence, it remains to show that C dominates $\mathbb{A}^1_k$ under f. Otherwise, $f(C)$ would be finite and there might exist s_0 such that $Z(f_{X_1}, \dots, f_{X_n}, f - s_0)$ is infinite in some neighbourhood at the origin of $\mathbb{A}^n_k$. But this is a contradiction because, if $s_0 \neq 0$, then $f - s_0$ does not vanish in some neighbourhood of the origin and if $s_0 = 0$ it says that f does not have an isolated singularity at the origin.

Now, if $\mu_0(f) < \infty$ then the same argument used in Proposition 2.1.1 shows that f belongs to the ideal $\sqrt{J(f)}$ of $\mathcal{O}_{\mathbb{A}^n_k,0}$. Hence there exists a relation

$$B\, f^N = A_1 f_{X_1} + \cdots + A_n f_{X_n}, \quad \text{with } A_1, \dots, A_n, B \in k[X_1, \dots, X_n],\ B(0) \neq 0. \tag{2.1}$$

Notice that each fiber $f^{-1}(s)$, with $s \neq 0$ in the open neighbourhood $\mathbb{A}^n_k \setminus Z(B)$ of the origin, is smooth. Indeed, if $x \in \mathbb{A}^n_k \setminus Z(B)$ is a singular point of the fiber $f^{-1}(s)$, with

$s \neq 0$, then $f(x) = s$ and $f_{X_i}(x) = 0$ for each $i = 1, \dots, n$. On the other hand, since $B(x) \neq 0$ it follows from (2.1) that $s = f(x) = 0$, which is a contradiction. ■

2.2 Good equations and the invariant $\mu(\mathcal{O})$

As we have already seen, in positive characteristic, the usual definition of μ is somehow pathological in the sense that when a power series f has an isolated singularity ($\tau(f) < \infty$) its Milnor number $\mu(f)$ may be infinite or may change in the same contact class. It is not, therefore a geometric invariant of the singularity because contact equivalent power series define isomorphic singularities. In the complex case, however, $\mu(f)$ is a very important *topological* invariant in the case of isolated singularities which is therefore preserved under contact equivalence and is always finite. These last two properties are also true when we work over (algebraically closed) fields of characteristic zero as is shown in [B] as an application of the *Lefschetz's Principle* and using the validity of the result over $\mathbb{C}$ (see Prop. 5.2.1 and Prop. 5.3.1 in [B]). The purpose of this section is to give completely algebraic proofs of the these results (no topological arguments are involved) and to suggest what would be the best definition of a Milnor number for an isolated hypersurface singularity over an arbitrary algebraically closed field.

To begin with we recall a definition:

Definition 2.2.1. *Let $I \subset \mathcal{R}$ be an ideal. An element $g \in \mathcal{R}$ for which there exists $\ell \geqslant 1$ and $a_1, \dots, a_\ell \in \mathcal{R}$ with $a_i \in I^i$ satisfying*

$$g^\ell + a_1 g^{\ell-1} + \cdots + a_\ell = 0$$

is said to be integral over I. We denote the set of elements which are integral over I as $\overline{I}$. It is well known that $\overline{I}$ is an ideal of $\mathcal{R}$. Obviously we have

$$I \subset \overline{I} \subset \sqrt{I}, \quad I \subseteq I' \Rightarrow \overline{I} \subseteq \overline{I'} \quad \text{and} \quad \overline{\overline{I}} = \overline{I}.$$

We say that an ideal I is integrally closed in $\mathcal{R}$ if it satisfies $I = \overline{I}$. We refer the reader to [H-S] for details.

Example 2.2.2. *Every principal ideal, say $I = g\mathcal{R}$, is integrally closed inside $\mathcal{R}$. In fact, this is clear if $g = 0$ since $\mathcal{R}$ is a domain. If $g \neq 0$ and $h \in \mathcal{R}$ is integral over I then we can write*

$$h^\ell + b_1 g h^{\ell-1} + \cdots + b_\ell g^\ell = 0,$$

for some $b_i \in \mathcal{R}$*. This means that* $\frac{h}{g} \in K$ *is integral over* $\mathcal{R}$*, where* K *is the field of fractions of* $\mathcal{R}$*. But* $\mathcal{R}$ *is integrally closed inside* K*, hence* $\frac{h}{g} \in \mathcal{R}$ *implying that* $h \in g\mathcal{R} = I$*.*

Definition 2.2.3. *Let* $J \subseteq I$ *be ideals of* $\mathcal{R}$*. The ideal* J *is said to be a reduction of* I *if for some* $s \geqslant 0$ *an equality* $JI^s = I^{s+1}$ *holds. Observe that in this case* $\sqrt{I} = \sqrt{J}$*. A reduction of an ideal* I *is said to be a* minimal reduction *if it is minimal with respect to inclusion.*

The concepts of integral element over an ideal and of a reduction of an ideal are related by the following.

Remark 2.2.4. *For any element* $g \in \mathcal{R}$ *and any ideal* $I \subseteq \mathcal{R}$*, we have* $g \in \overline{I}$ *if and only if there exists an integer* s *such that* $(I + \langle g \rangle)^s = I(I + \langle g \rangle)^{s-1}$*. In other words,* $g \in \overline{I}$ *if and only if* I *is a reduction of* $I + \langle g \rangle$*.*

Proof: First suppose that $g \in \overline{I}$. Then an equation of integral dependence of g over I of degree s shows that $g^s \in I(I + \langle g \rangle)^{s-1}$ and hence that $(I + \langle g \rangle)^s = I(I + \langle g \rangle)^{s-1}$. Conversely, if $(I + \langle g \rangle)^s = I(I + \langle g \rangle)^{s-1}$ then $g^s = b_1 g^{s-1} + b_2 g^{s-2} + \cdots + b_{s-1} g + b_s$ for some $b_i \in I^i$, which can be easily rewritten into an equation of integral dependence of g over I. ■

In order to show that the preceding concepts fit well for the ideals $J(f)$ and $T(f) = J(f) + \langle f \rangle$ we are going to show (in characteristic zero) that $f \in \overline{J(f)}$. Precisely, in characteristic zero, we have the following result attributed to B. Teissier (see [Te]).

Theorem 2.2.5. *Let* k *be a field (non necessarily algebraically closed) of characteristic zero. If* $f \in \mathcal{R} = k[[X_1, \cdots, X_n]]$ *is a non invertible element, then we have* $f \in \overline{\langle X_1 f_{X_1}, \ldots, X_n f_{X_n} \rangle}$*. Hence we also have* $f \in \overline{\mathfrak{m}J(f)} \subset \overline{J(f)}$*.*

Proof: See [H-S], proof of Proposition 7.1.5. ■

Remark 2.2.6. *Since* $\overline{J(f)} \subseteq \sqrt{J(f)}$ *we see in the particular case of characteristic zero that*

$$\tau(f) < \infty \Longrightarrow \mu(f) < \infty$$

using our criterion given by Proposition 2.1.1 for finiteness of $\mu(f)$ *and Theorem 2.2.5. Observe that this proof is completely algebraic (avoiding typical topological methods over* $\mathbb{C}$*) and that it holds even if the field* k *is not algebraically closed.*

Also a consequence of the preceding Theorem 2.2.5 and Remark 2.2.4 is that in characteristic zero the ideal $J(f)$ is always a reduction of the ideal $T(f)$. This is far from being the case if our ground field k has positive characteristic, as shows the following example.

Example 2.2.7. *Let $f = Y^p + X^{p+1}$, $\operatorname{char} k = p > 0$. Then $J(f) = \langle X^p \rangle$ and hence, according to Example 2.2.2, $J(f) = \overline{J(f)}$. Then f is not integral over $J(f)$ because otherwise f would be a multiple of X^p. Hence $J(f)$ is not a reduction of $T(f) = J(f) + \langle f \rangle$. Alternatively, we can see this directly by observing that $J(f)$ is not $\mathfrak{m}$-primary, though $T(f)$ is.*

We denote by $e_0(I)$ the Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal I. This is always a non-negative integer. We put $e_0(I) = \infty$ if $\sqrt{I} \subsetneq \mathfrak{m}$.

The next well known proposition describes, in arbitrary characteristic, the effect of the condition $f \in \overline{J(f)}$.

Proposition 2.2.8 ([N-R] and [R]). *Let $f \in \mathfrak{m} \subset \mathcal{R}$. The following conditions are equivalent:*

(i) $f \in \overline{J(f)}$;

(ii) $J(f)$ *is a reduction of* $T(f)$.

Furthermore, if $\tau(f) < \infty$, the two preceding conditions are equivalent to

(iii) $J(f)$ *is a minimal reduction of* $T(f)$;

(iv) $e_0(T(f)) = e_0(J(f))$.

Sketch of Proof: (i) $\Leftrightarrow$ (ii): Exactly the same as the proof of 2.2.4.

Now assume that $\tau(f) < \infty$.

(iii) $\Leftarrow$ (iv): This is a hard result proved originally by D. Rees in [R], Theorem 3.2 and we omit the proof. Another reference for this point (with an easier proof) is Theorem 11.3.1 in [H-S]. Notice that $\mathcal{R}$ is a formally equidimensional Noetherian local ring (or a *level local ring* in the terminology of Rees), that is, their minimal prime ideals have the same dimension.

(iii) $\Rightarrow$ (iv) Let $I = T(f)$ and $J = J(f)$. Since J is a reduction of I, by the assumption, we get, for some $s \geqslant 0$,

$$I^{s+1} = I^s J.$$

This, in turn, implies that

$$I^{s+m} = I^s J^m, \ \forall \ m \geqslant 0.$$

Hence, from $J \subseteq I$ we get

$$\dim(\mathcal{R}/J^{m+s}) \geqslant \dim(\mathcal{R}/I^{m+s}) \geqslant \dim(\mathcal{R}/J^m I^s) \geqslant \dim(\mathcal{R}/J^m), \ \forall m.$$

Since the Hilbert-Samuel multiplicity is the leading coefficient (normalized by $1/n!$) of the Hilbert-Samuel polynomial of the ideal we get $e_0(I) = e_0(J)$.

(iii) $\Rightarrow$ (ii) is clear;

(ii) $\Rightarrow$ (iii) Because if $\tau(f) < \infty$, $J(f)$ is a parameter ideal (see Proposition 2.1.1) and therefore $n = \dim \mathcal{R} = \dim_k \Big(J(f)/\mathfrak{m}J(f)\Big)$. This in turn implies that $J(f)$ is a minimal reduction of $T(f)$ (see for example [H-S], Corollary 8.3.5). ∎

Corollary 2.2.9. *Let $f \in \mathfrak{m}$ be such that $\tau(f) < \infty$. Then $f \in \overline{J(f)}$ if, and only if, $\mu(f) = e_0(T(f))$.*

Proof: From Proposition 2.2.8 one has that $f \in \overline{J(f)}$ if and only if $e_0(J(f)) = e_0(T(f))$. In this situation, one has that $J(f)$ is a parameter ideal. Since $\mathcal{R}$ is a Cohen-Macaulay ring it follows that $\mu(f) = e_0(J(f))$ (cf [Ma], Theorem 17.11), which gives the result. ∎

Corollary 2.2.10. *Let k be a field of characteristic zero and $f \in \mathfrak{m}$. Then $\mu(f)$ is invariant under contact equivalence.*

Proof: If $\tau(f) = \infty$ then $\mu(g) = \infty$, for every g in the same contact equivalence class and we are done. So we are restricted to the case $\tau(f) < \infty$. Since changing coordinates obviously does not change μ, we only need to show that $\mu(f) = \mu(uf)$ for every unit $u \in \mathcal{R}$. However, it is easy to see that $T(f) = T(uf)$, for every such u. In characteristic zero, both $J(f)$ and $J(uf)$ are reductions of $T(f)$, according to the previous proposition and Theorem 2.2.5. On the other hand, $\mathcal{R}$ is a regular (hence Cohen-Macaulay) local ring. Since $J(f)$ is a parameter ideal (Remark 2.2.6) we have $\mu(f) = e_0(J(f))$ (cf. [Ma], Theorem 17.11). Therefore,

$$\mu(f) = e_0(J(f)) = e_0(T(f)) = e_0(T(uf)) = e_0(J(uf)) = \mu(uf).$$

∎

Remark 2.2.11. *In arbitrary characteristic the inclusion* $J(f) \subset T(f)$ *implies that* $e_0(T(f)) \leqslant e_0(J(f))$ *(cf. [Ma], Formula 14.4), then*

$$e_0(T(f)) \leqslant e_0(J(f)) = \mu(f).$$

The inequality in the above remark may be strict, even if $\mu(f) < \infty$, as shows the following example.

Example 2.2.12. *Let* $\operatorname{char} k = p > 2$ *and* $f = X^p + X^{p+2} + Y^{p+2} \in k[[X, Y]]$*. As* $J(f) = \langle X^{p+1}, Y^{p+1} \rangle$*, we have that* $\mu(f) = (p+1)^2$ *and* $\tau(f) = p(p+1) < \infty$*. If* $g = (1+X)f$*, then an easy calculation with intersection indices shows that* $\mu(g) = I(g_X, g_Y) = p(p+1)$*, so*

$$e_0(T(f)) = e_0(T(g)) \leqslant \mu(g) = p(p+1) < (p+1)^2 = \mu(f).$$

It follows from the preceding discussion that the importance of the Jacobian ideal of a hypersurface singularity in characteristic zero is due to the fact that it is a reduction of the Tjurina ideal, which is the ideal that carries all the information about the singular point. In this situation, one has that $e_0(T(f)) = \mu(f)$ (we do not exclude the case in which the singularity is non-isolated accordingly to our convention that $e_0(I) = \infty$ if I is not $\mathfrak{m}$-primary), and this is why Milnor's number is full of meanings in characteristic zero.

This leads us to consider the *Milnor number of a hypersurface* $\mathcal{O} = \mathcal{O}_f$ as

$$\mu(\mathcal{O}) = e_0(T(f)).$$

Observe that this is an invariant of the algebroid hypersurface $\mathcal{O}$ and it is always finite if the singularity is isolated. Notice also that in characteristic zero one always has $\mu(\mathcal{O}) = \mu(f)$, for any equation f for $\mathcal{O}$.

Remark 2.2.13. *Proposition 2.2.8 gives a numerical criterion for testing whether* f *belongs or not to* $\overline{J(f)}$ *when the singularity is isolated. Example 2.2.12 shows that one may have* $f \in \sqrt{J(f)}$ *with* $f \notin \overline{J(f)}$*, since in this case* $\mu(f) > e_0(T(f)) = \mu(\mathcal{O}_f)$*.*

2.3 Variation of $\mu(uf)$ and computation of $\mu(\mathcal{O})$

We have seen that in characteristic zero, specially in the case of an isolated singularity $\mathcal{O} = \mathcal{O}_f$, that the multiplicity $e_0(T(f))$ may be computed as the codimension of $J(f)$ in $\mathcal{R}$ because, in that situation, $J(f)$ is a minimal reduction of $T(f)$ (see Proposition 2.2.8, Theorem 2.2.5 and Corollary 2.2.10). On the other hand, this is not always the case if the ground field has positive characteristic. Therefore, we are led to investigate whether $J(f)$ is a minimal reduction of $T(f)$ or, equivalently, whether $f \in \overline{J(f)}$ when $\tau(f) < \infty$. As a consequence of our discussion we will analyse the variation of $\mu(uf)$ when u varies in $\mathcal{R}^*$ and obtain a method for computing $\mu(\mathcal{O})$. More generally, we will search for (non necessarily minimal) reductions of $T(f)$.

Let $0 \neq f \in \mathfrak{m} \subset \mathcal{R}$ and let $\ell(T(f)) = \dim F_{T(f)}(\mathcal{R})$ denote the Krull dimension of the special fiber ring associated to to the blow-up of Spec$\mathcal{R}$ along $T(f)$:

$$F_{T(f)}(\mathcal{R}) := \bigoplus_{s \geqslant 0} \frac{T(f)^s}{\mathfrak{m}T(f)^s}.$$

This graded ring is called the *fiber cone* of $T(f)$. The integer $\ell(T(f))$ is called the *analytic spread* of the ideal $T(f)$ and it is known (see [H-S], Corollary 8.3.9) that

$$\operatorname{height} T(f) \leqslant \ell(T(f)) \leqslant \dim \mathcal{R} = n. \tag{2.2}$$

A special case of this is when $T(f)$ is $\mathfrak{m}$-primary: in this case clearly one has $\ell(T(f)) = \dim \mathcal{R} = n$.

Since $\mathcal{R}$ is a local ring with infinite residue field k and taking into account the bound (2.2), above, it is well known that for a fixed set of generators, not necessarily minimal, $f, f_{X_1}, \ldots, f_{X_n}$ of $T(f)$, if we take *sufficiently general* linear combinations

$$g_i = h_{0,i}\, f + \sum_{j=1}^{n} h_{j,i}\, f_{X_j}, \quad \text{where } h_{j,i} \in \mathcal{R}, \quad i = 1, \ldots, n, \tag{2.3}$$

then the ideal in $\mathcal{R}$ generated by $g_1, \ldots, g_n$ is a reduction of the ideal $T(f)$ (see [H-S] Theorem 8.6.6).

To make precise the preceding statement about the conditions on the $h_{j,i}$ to be sufficiently general, we will need the notion of null-forms.

A null-form (cf. [Ma], proof of Theorem 14.14) for the ideal $T(f)$ is a homogeneous polynomial $\varphi \in k[Y_0, \ldots, Y_n]$ of some degree s such that there exists $F \in \mathcal{R}[Y_0, \ldots, Y_n]$

homogeneous of degree s for which

$$F \equiv \varphi \bmod \mathfrak{m}\mathcal{R}[Y_0, \ldots, Y_n]$$

and $F(f, f_{X_1}, \ldots, f_{X_n}) \in \mathfrak{m}\, T(f)^s$. We denote by $\mathcal{N}_{T(f)}$ the homogeneous ideal in $k[Y_0, \ldots, Y_n]$ generated by all null-forms of $T(f)$.

Remark 2.3.1. *The ideal $\mathcal{N}_{T(f)}$ depends upon the generators $f, f_{X_1}, \ldots, f_{X_n}$ of $T(f)$. Nevertheless, as k-algebras one has from the construction that*

$$\frac{k[Y_0, \ldots, Y_n]}{\mathcal{N}_{T(f)}} \simeq \bigoplus_{s \geqslant 0} \frac{T(f)^s}{\mathfrak{m} T(f)^s} = F_{T(f)}(\mathcal{R}).$$

This implies that the projective zero set $Z(\mathcal{N}_{T(f)})$ in $\mathbb{P}^n_k$ is of dimension $\ell(T(f)) - 1 \leqslant n-1$. In particular, $\mathcal{N}_{T(f)} \neq \langle 0 \rangle$.

Example 2.3.2. *Recall Example 1.1.1, where $f = Y^p + X^{p+1} \in k[[X, Y]]$ and $\operatorname{char} k = p$. Since the polynomial $Y_2 \in k[Y_0, Y_1, Y_2]$ vanishes when evaluated at (f, f_X, f_Y), we have that $Y_2 \in \mathcal{N}_{T(f)}$. So, $Z(\mathcal{N}_{T(f)}) \subset Z(Y_2)$, and since $\dim(Z(\mathcal{N}_{T(f)})) = \dim(Z(Y_2))$ and $Z(Y_2)$ is irreducible, we have that $Z(\mathcal{N}_{T(f)}) = Z(Y_2)$. But this last equality, together with $Y_2 \in \mathcal{N}_{T(f)}$, imply that $\mathcal{N}_{T(f)} = \langle Y_2 \rangle$.*

Example 2.3.3. *Going back to Example 1.1.3, if $f = X^2Y + Y^2X \in k[[X, Y]]$, where $\operatorname{char} k = 3$, we have that $Y_0(Y_1 + Y_2) \in \mathcal{N}_{T(f)}$, since*

$$f(f_X + f_Y) = -XY(Y + X)(Y - X)^2 = (Y + X) f_X f_Y \in \mathfrak{m}\, T(f)^2.$$

Now, because $Y_0, Y_1 + Y_2 \notin \mathcal{N}_{T(f)}$, it follows that $\mathcal{N}_{T(f)}$ is not a prime ideal.

Remark 2.3.4. *For a computational routine to test whether a homogeneous polynomial is or not a null-form of $T(f)$ with worked examples see the Appendix A.*

Given $f \in \mathfrak{m}$, in order to have $e_0(T(f)) = \mu(uf)$, for some unit u, we must find $u \in \mathcal{R}^*$ such that $uf \in \overline{J(uf)}$. We will show in what follows that this is so for general units.

We will need the following result due to Northcott and Rees (cf. [N-R], or [H-S], proof of Theorem 8.6.6):

The ideal $\langle g_1, \ldots, g_n \rangle$, where the g_i's are as in (2.3) is a reduction of the ideal

$T(f)$ if and only if the linear forms

$$\ell_i = \sum_{j=0}^{n} a_{j,i} Y_j, \quad i = 1, \ldots, n,$$

where $a_{j,i} \in k$ and $a_{j,i} \equiv h_{j,i} \bmod \mathfrak{m}$, are such that the ideal $\mathcal{N}_{T(f)} + \langle \ell_1, \ldots, \ell_n \rangle$ is $\langle Y_0, \ldots, Y_n \rangle$-primary, that is, $Z(\mathcal{N}_{T(f)} + \langle \ell_1, \ldots, \ell_n \rangle) = \{0\} \subset \mathbb{A}_k^{n+1}$.

Theorem 2.3.5. *Let $f \in \mathfrak{m}$ and $u = \alpha_0 + \alpha_1 X_1 + \cdots + \alpha_n X_n + \text{hot}$, with $\alpha_0 \neq 0$ be a unit in $\mathcal{R}^*$. We have that $uf \in \overline{J(uf)}$ if and only if there exists $G \in \mathcal{N}_{T(f)}$ such that $G(\alpha_0, -\alpha_1, \ldots, -\alpha_n) \neq 0$. In particular, this holds for a generic $(\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_k^n$.*

Proof: If $g = uf$, then

$$g_{X_i} = u_{X_i} f + u f_{X_i}, \quad i = 1, \ldots, n,$$

with associated linear forms

$$\ell_i = \alpha_i Y_0 + \alpha_0 Y_i, \quad i = 1, \ldots, n.$$

We then have

$$Z(\mathcal{N}_{T(f)} + \langle \ell_1, \ldots, \ell_n \rangle) =$$

$$Z\left(\left\langle \left(\frac{Y_0}{\alpha_0}\right)^{\deg(G)} G(\alpha_0, -\alpha_1, \ldots, -\alpha_n),\ \alpha_1 Y_0 + \alpha_0 Y_1, \ldots, \alpha_n Y_0 + \alpha_0 Y_n;\ G \in \mathcal{N}_{T(f)} \setminus \{0\} \right\rangle\right).$$

Since $uf \in \overline{J(uf)}$ if and only if $J(uf) = \langle g_{X_1}, \ldots, g_{X_n} \rangle$ is a reduction of $T(uf) = T(f)$, then from the Northcott and Rees Theorem above mentioned, this happens if and only if $Z(\mathcal{N}_{T(f)} + \langle \ell_1, \ldots, \ell_n \rangle) = \{0\}$. This, in turn, happens if and only if for some $G \in \mathcal{N}_{T(f)}$, one has $G(\alpha_0, -\alpha_1, \ldots, -\alpha_n) \neq 0$. ∎

The above theorem shows that if u is a general unit, in the sense that it has a general linear term, then $J(uf)$ is a reduction of $T(uf)$, and so, $uf \in \overline{J(uf)}$, which in turn implies that

$$\mu(uf) = e_0(T(uf)) = \mu(\mathcal{O}_f).$$

Remark 2.3.6. *In arbitrary characteristic, the fact that $\mathcal{R}$ is a regular local ring implies that if $I \subset \mathcal{R}$ is an ideal generated by ℓ elements then, for all $m \geqslant 0$,*

$$\overline{I^{m+\ell}} \subseteq I^{m+1}.$$

For a proof of this result we refer the reader to [H-S], Theorem 13.3.3 (or [L-S].)
If $f \in \overline{J(f)}$*, this implies that* $\overline{J(f)} = \overline{T(f)}$ *and, consequently,* $\overline{J(f)^{\,a}} = \overline{T(f)^{\,a}}$, $\forall\, a \geqslant 0$ *(see [H-S], Corollary 1.2.5, for instance). Consequently, using the preceding result we obtain,* $\forall\, m \geqslant 0$,

$$\overline{T(f)^{n+m}} = \overline{J(f)^{n+m}} \subset J(f)^{m+1}.$$

For $m = 0$ *this gives*

$$f^n \in T(f)^n \subset \overline{T(f)^n} \subset J(f).$$

This is a famous theorem of Briançon and Skoda (see [BS]).

Theorem 2.3.5 also allows us to give the following interpretation for $\mu(O)$.

Corollary 2.3.7. *Let* $f \in \mathfrak{m}$ *be such that* $\tau(f) < \infty$*. Then* $\mu(\mathcal{O}_f) = \min\{\mu(uf),\ u \in \mathcal{R}^*\}$ *and* $\mu(f) = \mu(\mathcal{O}_f)$ *if and only if* $f \in \overline{J(f)}$*.*

Proof: From Remark 2.2.11 we know that $\mu(\mathcal{O}_f) = \mu(\mathcal{O}_{uf}) \leqslant \mu(uf)$, for all $u \in \mathcal{R}^*$. According to the last theorem there is a unit v such that $vf \in \overline{J(vf)}$, hence $\mu(\mathcal{O}_f) = \mu(vf)$, proving the first assertion. The second assertion follows immediately from Proposition 2.2.8. ■

Example 2.3.8. *Remark 2.3.6 shows that*

$$f \in \overline{J(f)} \Longrightarrow f^n \in J(f)$$

but the preceding Corollary allows one to see that in characteristic $p > 0$ *the converse is false.*

Indeed, consider $f = Y^3 + X^7 + X^6Y \in k[[X,Y]] = \mathcal{R}$ *where* k *is an algebraically closed field of characteristic* $p = 7$*. A computation shows that* $\tau(f) = 12$ *and* $\mu(f) = 16$*, but* $\mu(\mathcal{O}_f) = 14$ *as one can check observing that* $Y_1^2 \in \mathcal{N}_{T(f)} \setminus 0$*. Therefore* $f \notin \overline{J(f)}$*. However* $f^2 \in J(f)$ *because*

$$f^2 \equiv Y^6 + X^{14} \equiv (5Y^4 + X^8) f_Y \bmod f_X.$$

We also have the following result.

Theorem 2.3.9. *Let* $f \in \mathfrak{m}$ *be such that* $\tau(f) < \infty$*. The following three statements are equivalent.*

(i) $\mu(uf) = \mu(\mathcal{O}_f)$ *for every unit* $u \in \mathcal{R}$*;*

(ii) $Z(\mathcal{N}_{T(f)}) = Z(Y_0)$;

(iii) $f^\ell \in \mathfrak{m}T(f)^\ell$, *for some* $\ell \geqslant 1$.

Proof: $(i \Rightarrow ii)$ If $\mu(uf) = \mu(\mathcal{O}_f)$ for every unit u, then $Z(\mathcal{N}_{T(f)}) \cap \{Y_0 \neq 0\} = \emptyset$. Otherwise, if $(1 : \beta_1 : \cdots : \beta_n)$ is in this set, then we would have $G(1, \beta_1, \ldots, \beta_n) = 0$, for every $G \in \mathcal{N}_{T(f)}$, hence $u = 1 - \beta_1 X_1 - \cdots - \beta_n X_n$ would be such that $\mu(uf) > \mu(\mathcal{O}_f)$, contradicting the hypothesis.

Therefore, $Z(\mathcal{N}_{T(f)}) \subseteq Z(Y_0)$, which implies the equality by comparing dimensions and by the irreducibility of $Z(Y_0)$.
$(ii \Rightarrow iii)$ If $Z(\mathcal{N}_{T(f)}) = Z(Y_0)$ then $\sqrt{\mathcal{N}_{T(f)}} = \sqrt{\langle Y_0 \rangle} = \langle Y_0 \rangle$. Hence, there exists a positive integer ℓ such that $G = Y_0^\ell \in N_{T(f)}$. In other words, $f^\ell \in \mathfrak{m}T(f)^\ell$.
$(iii \Rightarrow i)$ If for some ℓ, one has $f^\ell \in \mathfrak{m}T(f)^\ell$, then $G = Y_0^\ell \in \mathcal{N}_{T(f)} \setminus \{0\}$. Let $u = \alpha_0 + \alpha_1 X_1 + \cdots + \alpha_n X_n + hot$ be a unit, then $G(\alpha_0, -\alpha_1, \ldots, -\alpha_n) = \alpha_0^\ell \neq 0$. It follows that $uf \in \overline{J(uf)}$ and, therefore, $\mu(uf) = \mu(\mathcal{O}_f)$. ■

Remark 2.3.10. *Without the assumption $\tau(f) < \infty$ we have the following weaker result:*

$$Z(\mathcal{N}_{T(f)}) \subseteq Z(Y_0) \Longleftrightarrow f^\ell \in \mathfrak{m}T(f)^\ell, \text{ for some } \ell \geqslant 1.$$

The proof follows the same lines of the proof of the preceding result.

Corollary 2.3.11. *Suppose $p = \operatorname{char} k = 0$ and let $f \in \mathfrak{m} \setminus \{0\}$ with $\tau(f) < \infty$. Then there exists $\ell \geqslant 1$ such that $f^\ell \in \mathfrak{m}T(f)^\ell$.*

Proof: For $p = 0$ we know that $\mu(\mathcal{O}_f) = \mu(f) = \mu(uf)$ for every invertible u, hence we may use the preceding theorem. ■

Remark 2.3.12 (cf. [Ga]). *The preceding corollary can be derived (without the assumption $\tau(f) < \infty$) from the fact that $f \in \overline{\mathfrak{m}\, J(f)}$, when $p = 0$, as we have already seen (see Theorem 2.2.5). Indeed, consider an equation of integral dependence of f over $\mathfrak{m}\, J(f)$:*

$$f^\ell + a_1 f^{\ell-1} + \cdots + a_\ell = 0,$$

with $a_i \in (\mathfrak{m}\, J(f))^i$. Hence, for each $i \geqslant 1$,

$$f^{\ell-i} a_i \in \mathfrak{m}^i\, f^{\ell-i}\, J(f)^i \subset \mathfrak{m}^i\, T(f)^\ell \subset \mathfrak{m}\, T(f)^\ell$$

and we conclude since $f^\ell = -\sum_i a_i f^{\ell-i}$.

However, if $p > 0$, one may try to produce an example of a power series f which satisfies the equivalent conditions of Theorem 2.3.9, but such that $f \notin \overline{\mathfrak{m}\, J(f)}$. We were not able to give explicitly this example with our methods. Therefore, this remains an open point.

If f is such that $\mu(uf)$ is independent of the unit u, that is, when $\mu(f)$ is invariant under contact equivalence, we will say that $\mathcal{O}_f$ is *μ-stable.* Of course this concept is relevant only in the case that $\tau(f) < \infty$ because otherwise $\mu(uf) = \infty$ for every unit $u \in \mathcal{R}$ and $\mathcal{O}_f$ is μ-stable trivially.

The third condition in Theorem 2.3.9 may help to decide whether a given hypersurface singularity $\mathcal{O}$ is or not μ-stable as we can see in the following examples. However, in order to have this condition as a computational method, we need to bound ℓ and this seems to be hard to do.

Example 2.3.13. *Let $f \in k[X_1, \ldots, X_n]$ be a quasi-homogeneous polynomial of degree d and $\operatorname{char} k = p > 0$. If $p \nmid d$, then $\mathcal{O}_f$ is μ-stable. Indeed, there are integers $d_1, \ldots, d_n$ such that*

$$df = d_1 X_1 f_{X_1} + \cdots + d_n X_n f_{X_n},$$

which, since $p \nmid d$, implies that $f \in \mathfrak{m}\, T(f)$, so by Theorem 2.3.9, $\mathcal{O}_f$ is μ-stable.

Proposition 2.3.14. *Let $f = G_1^p + h.o.t. \in \mathfrak{m} \subset \mathcal{R}$, where G_1 is a homogeneous polynomial and $\operatorname{char} k = p$. If $\mathcal{O}_f$ is isolated then it is not μ-stable.*

Proof: We will show that $f^\ell \notin \mathfrak{m}\, T(f)^\ell$, for all $\ell \in \mathbb{N}$.

Suppose $\deg G_1 = a$. We can write $f = G_1^p + H$, where $H \in \mathfrak{m}^{ap+1}$. It follows that $f_{X_i} = H_{X_i}$ and $\operatorname{mult}(f_{X_i}) = \operatorname{mult}(H_{X_i}) \geqslant ap$, for all $i = 1, \ldots, n$. Now, we have

$$T(f)^\ell = \langle f^{\alpha_0} f_{X_1}^{\alpha_1} \cdots f_{X_n}^{\alpha_n},\ \alpha_0 + \cdots + \alpha_n = \ell \rangle,$$

and

$$\operatorname{mult}(f^{\alpha_0} f_{X_1}^{\alpha_1} \cdots f_{X_n}^{\alpha_n}) = \operatorname{mult}(f^{\alpha_0} H_{X_1}^{\alpha_1} \cdots H_{X_n}^{\alpha_n}) \geqslant \sum_{i=0}^{n} \alpha_i\, a\, p = \ell\, a\, p.$$

This implies that $\operatorname{mult}(h) \geqslant \ell ap + 1$, for all $h \in \mathfrak{m}\, T(f)^\ell$; and since $\operatorname{mult}(f^\ell) = \ell ap$, it follows that $f^\ell \notin \mathfrak{m}\, T(f)^\ell$, for all $\ell \in \mathbb{N}$. ∎

The above proposition has the following corollary:

Corollary 2.3.15. *For $f \in k[[X, Y]]$ irreducible, where $\operatorname{char} k = p$, one has*

$$\mathcal{O}_f \text{ is } \mu - stable \ \Rightarrow\ p \nmid \operatorname{mult}(f).$$

Proof: This is a consequence of the fact that such an irreducible f has the form

$$f = L^{\mathrm{mult}(f)} + h.o.t.$$

where L is a linear form. This is, in turn, an easy consequence of Hensel's Lemma. Applying the preceding proposition we get the result. ∎

The converse of the above corollary is not true, as one may see from the following example:

Example 2.3.16. *Let $f = Y^3 - X^{11}$ where $\operatorname{char} k = 11$. Then $p \nmid \mathrm{mult}(f)$ but $\mathcal{O}_f$ is not μ-stable, since $\mu(f) = \infty$ and $\mu((1+X)f) = 22$.*

Notice that whether $\mathcal{O}_f$ is μ-stable, or not, depends upon the characteristic p of the ground field. For example, when $p = 5$ the same f as above is quasi-homogeneous of degree $d = 33$ which is not divisible by p. Hence it defines a μ-stable singularity $\mathcal{O}_f$.

Remark 2.3.17. *For all $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, that is, for all smooth hypersurface germs, one has μ-stability since in this case $T(f) = \mathcal{R}$ so that $f \in \mathfrak{m}T(f)$. Alternatively, in such case one has $\mu(f) = \mu(\mathcal{O}_f) = 0$.*

Also, we give an example to show that μ-stability is not preserved by blowing-up.

Example 2.3.18. *Let $\operatorname{char} k = 2$, and $f = Y^3 + X^5 \in k[[X,Y]]$. We have that f is μ-stable, since f is quasi-homogeneous of degree 15, not divisible by 2, but its strict transform $f^{(1)} = v^3 + u^2$ is not μ-stable (cf. Proposition 2.3.14).*

2.4 Levinson's Preparation Lemma

The Weierstrass Preparation Theorem plays an important role in the study of hypersurface singularities, since it transforms a power series into a polynomial in one of the variables. This process is not appropriate for studying Milnor's number in positive characteristic, since it involves the multiplication of the power series by a unit and this affects the Milnor number. So, we will need a preparation that involves only coordinate changes and this will be done using a result due to N. Levinson (cf. [Le]) over $\mathbb{C}$, which we adapt over arbitrary algebraically closed fields.

Theorem 2.4.1. *Let $0 \neq f(\underline{X}) \in \mathfrak{m} \subset \mathcal{R} = k[[X_1, \ldots, X_n]] = k[[\underline{X}]]$, where k is an algebraically closed field of characteristic $p \geq 0$. Write*

$$f = \sum_I \alpha_I \underline{X}^I$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, and suppose that f contains for some integer $r > 1$ a monomial X_n^r with nonzero coefficient. If r is minimal with this property and p does not divide r, then there exists a change of coordinates φ preserving $k[[\underline{X}']] = k[[X_1, \ldots, X_{n-1}]]$ induced by $X_n \mapsto X_n + X_n^2\, G(\underline{X})$, (for some element $G(\underline{X}) \in \mathcal{R}$) which transforms f into

$$\varphi(f) = A_r(\underline{X}')\, X_n^r + A_{r-1}(\underline{X}')\, X_n^{r-1} + \cdots + A_1(\underline{X}')\, X_n + A_0(\underline{X}'),$$

where $A_i(\underline{X}') \in k[[\underline{X}']]$ for every i and

$$A_{r-1}(0) = \cdots = A_1(0) = A_0(0) = 0, \quad A_r(0) \neq 0.$$

Proof: Since the announced φ does not change $\underline{X}'$ and since we can always take $A_0(\underline{X}') = f(\underline{X}', 0)$ there is no loss of generality in assuming $f(\underline{X}', 0) = 0 \in k[[\underline{X}']]$. Also, after dividing by a constant we can assume $\alpha_{(0,\ldots,r)} = 1$. Set $Y = X_n$. Therefore we have a writing

$$f(\underline{X}', Y) = \sum_{\ell=1}^{n-1} X_\ell^{a_\ell} \Big(\sum_{j=1}^{r} b_{j,\ell}(\underline{X}')\, Y^j \Big) + Y^r + Y^{r+1} \phi_0(\underline{X}', Y),$$

where, from the minimality of r, $a_1, \ldots, a_{n-1}$ are positive integers and $\phi_0 \in \mathcal{R}$. The rough idea of the proof is to eliminate the term $Y^{r+1}\phi_0(\underline{X}', Y)$ using a (convergent) sequence of coordinate changes.

Since $p \nmid r$ we can set

$$Y_1 := Y\,(1 + Y\phi_0)^{1/r} = Y \Big(1 + \frac{1}{r} Y \phi_0 + \cdots \Big).$$

We see that $\underline{X}'$ and Y_1 have k-linearly independent principal parts so that the change given by $Y \mapsto Y_1$ and which is the identity in $k[[\underline{X}']]$, is an invertible one. It allows us to write

$$Y = Y_1 + Y_1^2\, \psi_1(\underline{X}', Y_1)$$

for some $\psi_1 \in k[[\underline{X}', Y_1]] = k[[\underline{X}', Y]] = k[[\underline{X}]] = \mathcal{R}$.

By the definition of Y_1 we have

$$\begin{aligned} f(\underline{X}', Y) &= \sum_{\ell=1}^{n-1} X_\ell^{a_\ell} \Big(\sum_{j=1}^{r} b_{j,\ell}(\underline{X}')\,(Y_1 + Y_1^2\,\psi_1)^j \Big) + Y_1^r \\ &= \sum_{\ell=1}^{n-1} X_\ell^{a_\ell} \Big(\sum_{j=1}^{r} b_{j,\ell}^{(1)}(\underline{X}')\,Y_1^j \Big) + Y_1^r + Y_1^{r+1} \Big(\sum_{\ell=1}^{n-1} X_\ell^{a_\ell}\,\xi_{1,\ell}(\underline{X}', Y_1) \Big) \\ &= f_1(\underline{X}', Y_1), \end{aligned}$$

for some $\xi_{1,\ell}, b_{1,\ell}^{(1)} \in \mathcal{R}$ obtained after expanding the powers $(Y_1 + Y_1^2\,\psi_1)^j$ and collecting out terms according to the powers of Y_1. Denote $\phi_1 = \sum_{\ell=1}^{n-1} X_\ell^{a_\ell}\,\xi_{1,\ell}(\underline{X}', Y_1)$ and observe that $\operatorname{mult} \phi_1 > 0$.

The idea now is to iterate this procedure: set

$$Y_2 := Y_1\,(1 + Y_1 \phi_1)^{1/r}.$$

By the same argument there is an inverse and we have

$$Y_1 = Y_2 + Y_2^2\,\psi_2(\underline{X}', Y_2)$$

for some $\psi_2 \in k[[\underline{X}', Y_2]] = k[[\underline{X}', Y_1]] = k[[\underline{X}]] = \mathcal{R}$. Observe that after raising the two preceding expressions to the power r we get

$$-Y_1^{r+1}\,\phi_1 = -Y_2^r + Y_1^r = \psi_2 \left(r Y_2^2\,Y_2^{r-1} + \cdots + Y_2^{2r}\,\psi_2^{r-1} \right)$$

and after cancelling Y_1^{r+1} we see that ψ_2 and ϕ_1 are associate elements in $\mathcal{R}$, because $r \in k^*$. Hence $\operatorname{mult} \psi_2 = \operatorname{mult} \phi_1 > 0$. Substituting Y_2 in the expression of $f_1(\underline{X}', Y_1)$ we get

$$\begin{aligned} f_1(\underline{X}', Y_1) &= \sum_{\ell=1}^{n-1} X_\ell^{a_\ell} \Big(\sum_{j=1}^{r} b_{j,\ell}^{(1)}(\underline{X}')\,(Y_2 + Y_2^2\,\psi_2)^j \Big) + Y_2^r \\ &= \sum_{\ell=1}^{n-1} X_\ell^{a_\ell} \Big(\sum_{j=1}^{r} b_{j,\ell}^{(2)}(\underline{X}')\,Y_2^j \Big) + Y_2^r + Y_2^{r+1} \Big(\sum_{\ell=1}^{n-1} X_\ell^{a_\ell}\,\xi_{2,\ell}(\underline{X}', Y_2) \Big) \\ &= f_2(\underline{X}', Y_2), \end{aligned}$$

for some $\xi_{2,\ell}, b_{2,\ell}^{(2)} \in \mathcal{R}$ again obtained after expanding the powers $(Y_2 + Y_2^2\,\psi_2)^j$ and collecting out terms according to the powers of Y_2. It follows that if we denote $\phi_2 = \sum_{\ell=1}^{n-1} X_\ell^{a_\ell}\,\xi_{2,\ell}(\underline{X}', Y_2)$, then $\operatorname{mult}\phi_2 > \operatorname{mult}\phi_1 = \operatorname{mult}\psi_2$, because ψ_2 divides ϕ_2.

In the next step

$$Y_3 := Y_2\,(1 + Y_2\phi_2)^{1/r}$$

and so on. At the N-th step we deduce that

$$\operatorname{mult}\Big(\phi_N(\underline{X}', Y_N)\Big) > \operatorname{mult}\Big(\phi_{N-1}(\underline{X}', Y_{N-1})\Big).$$

Hence the sequence $\phi_N(\underline{X}', Y_N) \to 0$. Notice that at the N-th stage

$$\begin{array}{rcl} Y_2^r & = & Y_1^r + Y_1^{r+1}\phi_1 \\ Y_3^r & = & Y_2^r + Y_2^{r+1}\phi_2 \\ \vdots & = & \vdots \\ Y_N^r & = & Y_{N-1}^r + Y_{N-1}^{r+1}\phi_{N-1}. \end{array}$$

Hence $Y_N^r = Y_1^r + (Y_1^{r+1}\phi_1 + \cdots + Y_{N-1}^{r+1}\phi_{N-1})$. Observe that, as $N \to \infty$,

$$Y_1^{r+1}\phi_1 + \cdots + Y_{N-1}^{r+1}\phi_{N-1} \to \phi \in \mathcal{R}$$

because $\operatorname{mult}\phi_i > \operatorname{mult}\phi_{i-1}$ for all i. Hence, $Y_\infty := \lim Y_N = (Y_1^r + \phi)^{1/r}$ induces a coordinate change φ that does the job. It follows that $\varphi(f)$ has the sought form.

■

CHAPTER 3

Irreducible plane curves

3.1 A fundamental formula

Let $\mathcal{O}$ be an irreducible singular and plane algebroid curve with maximal ideal $\mathfrak{m}$. Consider the integral closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in its field of fractions. We define the conductor ideal $\mathcal{C}(\mathcal{O}) := (\mathcal{O} : \overline{\mathcal{O}})$. This is the largest common ideal of $\mathcal{O}$ and $\overline{\mathcal{O}}$. As an ideal of $\overline{\mathcal{O}}$ we have

$$t^c\overline{\mathcal{O}} = \mathcal{C}(\mathcal{O}) \tag{3.1}$$

for some $c \in \mathbb{N}$. This number $c = c(\mathcal{O})$ is an invariant of $\mathcal{O}$ called the degree of the conductor.

Choose generators x, y of $\mathfrak{m}$ and a uniformizing parameter $t \in \overline{\mathcal{O}}$ such that the images $x(t)$ and $y(t)$ in $\overline{\mathcal{O}} \simeq k[[t]]$ are a primitive parametrization for any equation f of $\mathcal{O}$ determined by the kernel of the epimorphism $\mathcal{R} \to \mathcal{O}$ given by $X \mapsto x$ and $Y \mapsto y$, that is $k((x(t), y(t))) = k((t))$. We say that $z \in \mathfrak{m}$ is a *transversal* parameter for $\mathcal{O}$ if $v(z) = \min\{v(w) \,|w \in \mathfrak{m}\}$ where $v = \mathrm{ord}_t$ is the natural valuation of $\overline{\mathcal{O}}$, which coincides with the intersection index with f, in the sense that $v(g) = I(f, G)$, where $G \in \mathcal{R}$ is any representative of the residual class $g \in \mathcal{O}$. This minimum is called the multiplicity of $\mathcal{O}$ and is denoted $\mathrm{mult}(\mathcal{O})$. We also say that z is a *separable* parameter with respect to t if $z'(t) = \frac{dz}{dt} \neq 0$. For $h \in \mathcal{R}$, we will write h_x for $h_X(x, y) \in \mathcal{O}$ and similarly for h_y.

The following fundamental formula is classical and the oldest proof we have found of it is due to D. Gorenstein, which we present below.

Theorem 3.1.1. *([Go], Theorem 12) Let $\mathcal{O}$ be a plane, irreducible, algebroid curve. If x, y is any system of generators of $\mathfrak{m} \subset \mathcal{O}$ and f is any equation corresponding to the embedding determined by x and y, then*

$$v(f_y) = c + v(x').$$

Proof: We treat first the case in which x is both a transversal and separable parameter and the tangent line is Y. We will prove this by induction in the number of blowing-ups necessary to dessingularize the branch $\mathcal{O}$. If $\mathcal{O}$ is already non-singular then the result is obvious.

Since in this coordinate system the tangent cone of an equation f is Y^n ($n = \operatorname{mult}\mathcal{O}$) then, if $f^{(1)}$ is the induced equation of the strict transform, we have

$$X^n f^{(1)}(X,Z) = f(X,Y)$$

where $Y = XZ$. Differentiation with respect to Z in the last relation leads to

$$X^n f_Z^{(1)}(X,Z) = X f_Y(X,Y).$$

This relation modulo $\langle f \rangle$ gives, after cancelling x,

$$x^{n-1} f_z^{(1)} = f_y.$$

Hence

$$v(f_y) = v(x^{n-1}) + v(f_z^{(1)}) = n(n-1) + v(f_z^{(1)}).$$

By the induction hypothesis $v(f_z^{(1)}) = v(x') + c^{(1)}$. According to a well known formula (see for instance [He], formula (6.10)), $c = c^{(1)} + n(n-1)$ and that concludes the proof in this case.

In the remaining case, that is, when x, y is an arbitrary set of generators of $\mathfrak{m}$, it is clear that there exists a system of generators z, w of $\mathfrak{m}$ such that

$$x = az + bw, \qquad y = cz + dw$$

where $a, b, c, d \in k$ with $D = ad - bc \neq 0$ and, say, z is both a transversal and separable parameter of $\mathcal{O}$ and w is the tangent line. Then, for any equation $g \in \mathcal{R}$ corresponding to the embedding determined by z and w, we get

$$g_w = f_x b + f_y d.$$

It follows that

$$f_y D z' = (dx' - by')f_y = dx' f_y + bx' f_x = x' g_w$$

and, therefore, $v(x') + v(g_w) = v(z') + v(f_y)$. Since by the preceding case we have $v(g_w) = c + v(z')$ this leads to $v(f_y) = c + v(x')$ in this general case. ■

Remark 3.1.2. *Zariski gave in [Za] a different proof for the preceding formula when* $\operatorname{char} k = 0$ *using Puiseux expansions. Maybe for this reason this formula is known as Zariski's (or Teissier's) formula. We thank Karl-Otto Stöhr for pointing out to us the connection between Zariski's and Gorenstein's formulas, the latter given originally in terms of the value of the meromorphic differential* $\frac{dx}{f_y}$.

Remark 3.1.3. *Observe, in the case of plane curves singularities defined by a power series* f*, the following equality:*

$$\mu(\mathcal{O}) = e_0(T(f)) = \min\{I(g,h) \mid g,h \in T(f)\}.$$

Indeed, if g, $h \in T(f)$ *we have that* $\langle g,h\rangle \subseteq T(f)$ *so that* $e_0(T(f)) \leqslant e_0(\langle g,h\rangle) = I(g,h)$*. On the other hand, say by Theorem 2.3.5, there are elements* $g,h \in T(f)$ *such that* $e_0(T(f)) = I(g,h)$.

Remark 3.1.4. *Proposition 3.1.1 also allows us to bound the number of wild vanishing cycles of* $\mathcal{O} = \mathcal{O}_f$*, that is, the difference between the Milnor number* $\mu(\mathcal{O})$ *of an algebroid irreducible plane curve and its conductor degree* c *as follows: first, from Deligne's results in [De] [Théorème 2.4](see also [MH-W]) one always has* $\mu(f) \geqslant c$*. Then, according to our preceding Remark,*

$$\mu(\mathcal{O}) = \min\{I(g,h) \mid g,h \in T(f)\} \leqslant I(f, f_Y) = v(x') + c.$$

Hence, we get

$$0 \leqslant \mu(\mathcal{O}) - c \leqslant v(x').$$

Of course, this is useful only if x *is separable parameter of* $\mathcal{O}$*. In particular, if* $\operatorname{char} k \nmid n = \operatorname{mult}\mathcal{O}$ *and* x *is transversal then* $0 \leqslant \mu(\mathcal{O}) - c \leqslant \operatorname{mult}\mathcal{O} - 1$*. This upper bound can be attained: see Example 2.3.8, for instance.*

Example 3.1.5. *For plane irreducible curves with semigroup of values* $\langle 3, 11\rangle$ *the conductor is* $c = 20$. *For the curves with equations* $f = Y^3 - X^{11} + X^8 Y$ *and* $h = Y^3 - X^{11} + X^9 Y$ *in characteristic* $p = 3$ *we find*

$$I(f, f_Y) - c = 4 \neq n - 1$$

and

$$I(h, h_Y) - c = 7 \neq n - 1.$$

We will need another formulation for Theorem 3.1.1, that is quoted in the literature as *Delgado's Formula* (cf. [Ca] (Proposition 7.4.1)), proved over $\mathbb{C}$, which extends naturally to arbitrary algebraically closed fields.

To begin with, let $\mathcal{O}$ be an algebroid irreducible plane curve. Choose an equation so that $\mathcal{O} = \mathcal{O}_f$. For $g \in \mathcal{O}_f$, take a representative $G \in \mathcal{R}$ of g and define

$$[f, g] := f_x G_y - f_y G_x \in \mathcal{O} \text{ and } g'(t) := \frac{d}{dt} g(x(t), y(t)).$$

Notice that $[f, g]$ is well defined since it does not depend on the representative G of g.

With these notations we have

Corollary 3.1.6. *Let* $\mathcal{O}$ *be the local ring of a plane irreducible algebroid curve and fix* t *a uniformizing parameter for* $\mathcal{O}$. *Then, for every* $g \in \mathcal{O}$, *one has*

$$v([f, g]) = c + v(g'(t)).$$

Proof: Let $y(t)$ be such that $(x(t), y(t))$ is local a parametrization of $\mathcal{O}$.
From $f(x(t), y(t)) = 0$ we obtain $f_X(x(t), y(t))x' + f_Y(x(t), y(t))y' = 0$, hence it is easy to check the following identity

$$x'(f_Y g_X - f_X g_Y)(x(t), y(t)) = f_Y(x(t), y(t))(g'(x(t), y(t))).$$

We deduce the result computing the orders and using Proposition 3.1.1. ■

Remark 3.1.7. *Since* $v(f_x) = c + v(y') \geqslant c$ *and* $v(f_y) = c + v(x') \geqslant c$, *we conclude that*

$$\mathrm{j} = \langle f_x, f_y\rangle \subset \mathcal{C}(\mathcal{O}).$$

This in turn implies, in particular, that $\delta \leqslant \tau(\mathcal{O})$.

Corollary 3.1.8. *If $p = \operatorname{char} k$, then*

$$v([f,g]) \geqslant c + v(g) - 1,$$

with equality holding if and only if $p \nmid v(g)$.

Proof: Since $v(g) = v\Big(g(x(t), y(t))\Big)$, we have

$$v(g) - 1 \leqslant v\Big(g'(t)\Big),$$

with equality if and only if $p \nmid v(g)$. The preceding Corollary gives, therefore,

$$\begin{aligned} v([f,g]) &= c + v(g'(t)) \\ &\geqslant c + v(g) - 1, \end{aligned}$$

where equality holds if and only if $p \nmid v(g)$. ∎

3.2 Milnor number for plane branches

For the definitions and notation used in this section we refer to [He] where these notions are characteristic free. Let $f \in \mathfrak{m} \subset k[[X,Y]]$ be an irreducible power series, where k is an algebraically closed field of characteristic $p \geqslant 0$. In most of this and the next section we will work with the equation f instead of the local ring $\mathcal{O}_f$ by convenience of notation, though the results are valid for the algebroid curve $\mathcal{O}$. Let us denote, for instance, by $S(f) = \langle v_0, \ldots, v_g \rangle$ the semigroup of values of the branch $\mathcal{O}_f$, represented by its minimal set of generators. These semigroups have many special properties which we will use throughout this section and describe them briefly below.

Let us define $e_0 = v_0$ and denote by $e_i = \gcd\{v_0, \ldots, v_i\}$ and by $n_i = e_{i-1}/e_i$, $i = 1, \ldots, g$. The semigroup $S(f)$ is *strongly increasing*, which means that $v_{i+1} > n_i v_i$, for $i = 0, \ldots, g-1$, (cf. [He], (6.5)). This implies that the the sequence $v_0, \ldots, v_g$ is *nice*, which means that $n_i v_i \in \langle v_0, \ldots, v_{i-1} \rangle$, for $i = 1, \ldots, g$, (cf. [He], Proposition 7.9). This, in turn, implies that the semigroup $S(f)$ has a *conductor*, denoted by $c(f)$, which is the integer characterized by the following property: $c(f) - 1 \notin S(f)$ and $x \in S(f)$, for all $x \geqslant c(f)$ (of course $c(f)$ is exactly the degree c of the conductor as defined in (3.1)), and it is given by the formula (cf. [He], (7.1))

$$c(f) = \sum_{i=1}^{g} (n_i - 1)v_i - v_0 + 1.$$

The semigroup $S(f)$ is also symmetric (cf. [He] Proposition 7.7), that is,

$$\forall z \in \mathbb{N},\ z \in S(f) \iff c - 1 - z \notin S(f).$$

To deal with the positive characteristic situation, we introduce the following definition:

We call $S(f)$ a *tame semigroup* if p does not divide v_i for all $i \in \{0, \ldots, g\}$.

Recall that two plane branches over the complex numbers are *equisingular* if their semigroups of values coincide. We will keep this terminology even in the case of positive characteristic.

The following example will show that $\mu(\mathcal{O}_f)$ may be not constant in an equisingularity class of plane branches.

Example 3.2.1. *The curves given by* $f = Y^3 - X^{11}$ *and* $h = Y^3 - X^{11} + X^8Y$ *are equisingular with semigroup of values* $S = \langle 3, 11 \rangle$*, but in characteristic* 3*, one has* $\mu(\mathcal{O}_f) = \mu((1+Y)f) = 30$*, because* $Y_2 \in \mathcal{N}_{T(f)}$ *and* $\mu(\mathcal{O}_h) = \mu((1+X)h) = 24$*, because* $Y_1^3 \in \mathcal{N}_{T(h)}$*. Notice that in this case* S *is not tame.*

Remark 3.2.2. *For the above* h *one has* $\mu(h) = \infty$*,* $\mu(\mathcal{O}_h) = 24$ *and* $\tau(h) = 22$*. This shows that there is no isomorphism* φ *of* $k[[X,Y]]$ *and no* $H \in k[[X,Y]]$ *such that* $\varphi(h) = H(X, Y^3)$*, because, otherwise, we would get the contradiction*

$$24 = \mu(\mathcal{O}_h) = \mu(\mathcal{O}_{\varphi(h)}) = \mu(\mathcal{O}_{H(X,Y^3)}) = \tau(H(X,Y^3)) = \tau(h) = 22.$$

For a characterization of those $f \in \mathcal{R} = k[[X,Y]]$ *for which there exists an isomorphism* φ *of* $\mathcal{R}$ *such that* $\varphi(f) \in k[[X, Y^p]]$ *see Appendix B.*

The following is an example which shows that the μ-stability is not a character of an equisingularity class.

Example 3.2.3. *Let* $S = \langle 4, 6, 25 \rangle$ *be a strongly increasing semigroup with conductor* $c = 28$*. Consider the equisingularity class determined by* S *over a field of characteristic* $p = 5$*. If* $f = (Y^2 - X^3)^2 - YX^{11}$*, which belongs to this equisinsingularity class, we have that* $\mu(f) = 41$ *and* $\mu(\mathcal{O}_f) = 30$*, hence* $\mathcal{O}_f$ *is not* μ*-stable. But the equisingular curve* $\mathcal{O}_h$

where $h = (Y^2 - X^3 + X^2Y)^2 - YX^{11}$ *is* μ*-stable, since* $Y_0^3 \in \mathcal{N}_{T(h)}$*. In this case one has* $\mu(h) = \mu(\mathcal{O}_h) = 29$*. Notice that here, again,* S *is not tame.*

The aim of this part of the thesis is to prove the following result:

Theorem 3.2.4 (Main Theorem)**.** *If* $f \in \mathfrak{m}^2$ *is a plane branch singularity with* $S(f)$ *tame, then* $\mu(f) = \mu(\mathcal{O}_f) = c(f)$*. In particular,* $\mathcal{O}_f$ *is* μ*-stable.*

The proof we give of this theorem is based on the following theorem which was stated without a proof over the complex numbers in [Ja1], but proved in the unpublished work [Ja2]. Our proof, in arbitrary characteristic, is inspired by that work, which we suitably modified in order to make it work in the more general context we are considering.

Theorem 3.2.5 (Key Theorem)**.** *Let* $f \in \mathfrak{m}^2$ *be an irreducible Weierstrass polynomial such that* $S(f)$ *is tame. Then any family* $\mathcal{F}$ *of elements inside* $k[[X]][Y]$ *of degree in* Y *less than* $\operatorname{mult}(f)$ *such that*

$$\{I(f,h);\ h \in \mathcal{F}\} = S(f) \setminus\ (S(f) + c(f) - 1)$$

is a representative set of generators of the k*-vector space* $\mathcal{R}/J(f)$*.*

We postpone the proof of this theorem until the next section, since it is long and quite technical.

For the moment we observe here the following refinement of Theorem 2.4.1 for the case of plane branches.

Corollary 3.2.6. *Let* $f \in k[[X,Y]]$ *be irreducible where* k *is algebraically closed of characteristic* p*. If* $p \nmid n := \operatorname{mult}(f)$*, then there exists an automorphism* φ *of* $k[[X,Y]]$ *such that*

$$\varphi(f) = Y^n + B_{n-1}(X)Y^{n-1} + \cdots + B_1(X)Y + B_0(X),$$

where $B_i(X) \in k[[X]]$ *and* $\operatorname{mult}(B_{n-i}) > i$*, for all* $i = 1, \ldots, n$*.*

Proof: Since f is irreducible, we have that $f = L^n + hot$, where L is a linear form in X and Y. By changing coordinates, we may assume that f is as in the conclusion of Theorem 2.4.1. Now, since $p \nmid n$, we take an n-th root of $A_n(X)$ and perform the change of coordinates $Y \mapsto YA_n^{\frac{1}{n}}$ and $X \mapsto X$. So, after only changes of coordinates φ, we have that

$$\varphi(f) = Y^n + B_1(X)Y^{n-1} + \cdots + B_{n-1}(X)Y + B_n(X),$$

is a Weierstrass polynomial, that is, $\mathrm{mult}(B_{n-i}(X)) > i$, for $i = 1, \ldots, n$. ■

Remark 3.2.7. *The above corollary is similar to the Weierstrass Preparation Theorem, but without multiplication by units.*

Proof of Theorem 3.2.4: From Deligne's results in [De] (see also [MH-W]) one always has $\mu(f) \geqslant c(f)$.

Now, after a change of coordinates, that does not affect the result, we may assume that f is a Weierstrass polynomial. For every $\alpha \in S(f) \setminus (S(f)+c(f)-1)$, take an element $g \in k[[X,Y]]$ such that $I(f,g) = \alpha$ and after dividing it by f by means of the Weierstrass Division Theorem, we get in this way a family $\mathcal{F}$ as in Theorem 3.2.5.

Theorem 3.2.5 asserts that the residue classes of the elements in $\mathcal{F}$ generate $k[[X,Y]]/J(f)$, hence $\mu(f) \leqslant \#\left(S(f) \setminus (S(f) + c(f) - 1)\right)$. The result will then follow from the next lemma that asserts that the number in the right hand side of the inequality is just $c(f)$.

The μ-stability follows from the fact that for every invertible element u in $k[[X,Y]]$, both power series f and uf can be individually prepared to Weierstrass form by means of a change of coordinates that does not alter the semigroup, nor the Milnor numbers. Hence, $\mu(f) = c(f) = \mu(uf)$. ■

Lemma 3.2.8. $\#\left(S(f) \setminus (S(f) + c(f) - 1)\right) = c(f)$.

Proof: In fact, to every $i \in \{0, 1, \ldots, c(f)-1\}$ we associate $s_i \in S(f) \setminus (S(f)+c(f)-1)$ in the following way:

$$s_i = \begin{cases} i, & \text{if } i \in S(f) \\ i + c(f) - 1, & \text{if } i \notin S(f). \end{cases}$$

The map $i \mapsto s_i$ is injective since $S(f)$ is a symmetric semigroup. On the other hand, the map is surjective, because, given $j \in S(f) \setminus (S(f) + c(f) - 1)$, we have $j = s_j$ if $j \leqslant c(f) - 1$; otherwise, if $j = i + c(f) - 1$ for some $i > 0$, then again by the symmetry of $S(f)$, it follows that j does not belong to $S(f)$ and therefore $j = s_i$. ■

We believe that the converse of Theorem 3.2.4 is true, in the sense that if $\mu(f) = c(f)$, then $S(f)$ is a tame semigroup, or, equivalently, if p divides any of the minimal generators of $S(f)$, then $\mu(f) > c(f)$. If this is so, we would conclude from our result that if $\mu(f) = c(f)$, then $\mathcal{O}_f$ is μ-stable.

To reinforce our conjecture, observe that the result of [GB-P] proves it when $\mathrm{mult}(\mathcal{O}) = \mathrm{mult}(f) < p$. For another evidence of the validity of that converse we refer to

Appendix C of this work. The following example is a situation where the converse holds and is not covered by the result in [GB-P].

Example 3.2.9. *Let p be any prime number and n and m two relatively prime natural numbers such that $p \nmid n$, then all curves given by $f(X,Y) = Y^n - X^{mp}$ do not satisfy the condition $\mu(f) = c(f)$, since $\mu(f) = \infty$ and $c(f) = (n-1)(mp-1)$. So, for all $p < n$, we have examples for the converse of our result not covered by [GB-P].*

Anyway, the other possible converse of 3.2.4, namely, if f is μ-stable then $S(f)$ is tame, is not true, as one may see from the following example.

Example 3.2.10. *Let $f = (Y^2 - X^3 + X^2Y)^2 - X^{11}Y \in k[[X,Y]]$, where $\operatorname{char} k = 5$. Since $f^3 \in \mathfrak{m}T(f)^3$ (verified with Singular), then $\mathcal{O}_f$ is μ-stable, but its semigroup of values $S(f) = \langle 4, 6, 25 \rangle$ is not tame.*

3.3 Proof of the Key Theorem

We start with an auxiliary result. Let $f \in \mathcal{R}$ be an irreducible Weierstrass polynomial in Y of degree $n = v_0$, where $S(f) = \langle v_0, \dots, v_g \rangle$, $I(f,X) = v_0$ and $I(f,Y) = v_1$.

Consider the $k[[X]]$-submodule V_{n-1} of $k[[X,Y]]$ generated by $1, Y, \dots, Y^{n-1}$, and let $h_0 = 1, h_1, \dots, h_{n-1}$ be polynomials in Y such that

$$V_{n-1} = k[[X]] \oplus k[[X]]h_1 \oplus \cdots \oplus k[[X]]h_{n-1},$$

and their residual classes y_i are the Apéry generators of $\mathcal{O}_f$ as a free $k[[X]]$-module (cf. [He] Proposition 6.18).

The natural numbers $a_i = v(y_i) = I(f, h_i)$, $i = 0, \dots, n-1$, form the Apéry sequence of $S(f)$, so they are such that $0 = a_0 < a_1 < \cdots < a_{n-1}$ and $a_i \not\equiv a_j \bmod n$ for $i \neq j$ (cf. [He] Proposition 6.21).

We have the following result.

Proposition 3.3.1. *Let I be an $\mathfrak{m}$-primary ideal of $\mathcal{R}$ and $h \in V_{n-1}$. If $I(f,h) >> 0$ then $h \in I$.*

Proof: Since the ideal I is $\mathfrak{m}$-primary, there exists a natural number l such that $\mathfrak{m}^l \subset I$.

Now, write $h = b_0 + b_1h_1 + \cdots + b_{n-1}h_{n-1}$, with $b_i \in k[[X]]$, for all i. Since $I(f, b_i) \equiv 0 \bmod n$, $I(f, h_i) = a_i$ and $a_i \not\equiv a_j \bmod n$, for $i, j = 0, \dots, n-1$, with $i \neq j$, we have that

$$I(f,h) = \min_i\{I(f,b_i) + a_i\} \leqslant \min_i\{I(f,b_i)\} + a_{n-1}.$$

Hence, $I(f,h) >> 0$ implies that for a given natural number l we have that $\min_j\{I(f,b_j)\} > lv_0$, hence, $h \in \mathfrak{m}^l \subset I$, as we wanted to show. ∎

Remark 3.3.2. *If f is a Weierstrass polynomial in Y of degree v_0 and $p \nmid v_0$, we proved in Corollary 3.1.6 that*

$$I(f,[f,g]) \geqslant I(f,g) + c(f) - 1,$$

with equality if and only if $p \nmid I(f,g)$, where here $[f,g] := f_X g_Y - f_Y g_X$.

Now, under the assumptions that f is a Weierstrass polynomial in Y with $S(f) = \langle v_0,\ldots,v_g\rangle$ and $p \nmid v_0$, one may associate the Abhyankar-Moh approximate roots (cf. [A-M], paragraphs 6 or 7), which are irreducible Weierstrass polynomials $f_{-1} = X, f_0 = Y,\ldots,f_g = f$ such that, for each j, one has $S(f_j) = \langle \frac{v_0}{e_j},\ldots,\frac{v_j}{e_j}\rangle$ and $I(f,f_j) = v_{j+1}$, satisfying a relation, where deg stands for degree as polynomial in Y,

$$f_j = f_{j-1}^{n_j} - \sum_{i=0}^{n_j-2} a_{ij} f_{j-1}^i,$$

where a_{ij} are polynomials in Y of degree less than $\deg(f_{j-1}) = v_0/e_{j-1}$ for $j = 0,\ldots,g$.

So, from Remark 3.3.2 we have that

$$I(f,[f,f_{j-1}]) \geqslant v_j + c(f) - 1, \quad \text{with equality if and only if } p \nmid v_j. \tag{3.2}$$

This implies that if $p \nmid v_0 v_1 \cdots v_g$, then $S(f)^* + c(f) - 1 \subset \nu(J(f)) := \{I(f,h) \mid h \in J(f)\}$, where $S(f)^* = S(f) \setminus \{0\}$.

The key result to prove Theorem 3.2.5 is Proposition 3.3.3 below that will allow us to construct elements in $J(f) \cap V_{n-1}$ whose intersection multiplicity with f sweep the set $S(f)^* + c(f) - 1$.

Proposition 3.3.3. *Let $f \in k[[X,Y]]$ be an irreducible Weierstrass polynomial in Y of degree v_0, where k is an algebraically closed field of characteristic $p \geqslant 0$. Let $S(f) = \langle v_0,\ldots,v_g\rangle$ and suppose that $p \nmid v_0 v_1 \cdots v_g$. Given $s \in S(f)^*$, there exists $q_s \in J(f)$, polynomial in Y, satisfying*

(i) $\deg q_s < \deg f = v_0$;

(ii) $I(f,q_s) = s + c(f) - 1$.

Proof: The proof will be by induction on the genus g of f. We will construct step by step the polynomial q_s which will be of the form $q_s = q_{f,s} = \sum_i P_i[f, f_{j_i}]$ (an infinite sum, possibly) where each f_{j_i} is an approximate root of f and the P_i are monomials in the approximate roots of f satisfying the following conditions:

$$\begin{cases} I(f, P_1 f_{j_1}) = s; \\ I(f, P_i f_{j_i}) > s, & \text{if } i \neq 1; \\ \deg f_{j_i} P_i < \deg f, & \text{for all } i. \end{cases} \tag{3.3}$$

If $g = 0$, we have $f = Y$, so $J(f) = \mathcal{R}$. Given $s \in \mathbb{N}^* = S(f)^*$, set

$$q_{f,s} := X^{s-1}[f, X].$$

It is easy to check that $q_{f,s}$ satisfies (3.3) and the conclusion of the proposition.

Inductively, we assume that the construction was carried on for branches of genus $g-1$. Consider the approximate root f_{g-1} of f of genus $g-1$. Since $e_{g-1} = n_g$ and $n_g v_g \in \langle v_0, \ldots, v_{g-1} \rangle$, we have

$$S(f) = \langle v_0, \ldots, v_g \rangle \subset \left\langle \frac{v_0}{n_g}, \ldots, \frac{v_{g-1}}{n_g} \right\rangle = S(f_{g-1}).$$

For $t \in S(f_{g-1})^*$, the inductive hypothesis guarantees the existence of a Y-polynomial

$$q_{f_{g-1},t} = \sum_i P_i[f_{g-1}, f_{j_i}],$$

where each f_{j_i} is one of the approximate roots $f_{-1}, f_0, \ldots, f_{g-2}$ and P_i are monomials in these approximate roots satisfying (3.3) and the conclusion of the proposition, with f_{g-1} and v_0/e_{g-1} replacing f and v_0, respectively. Using this $q_{f_{g-1},t}$ we introduce the following *auxiliary polynomial*

$$\tilde{q}_{f_{g-1},t} := \sum_i P_i[f, f_{j_i}].$$

To begin with, we will estimate the degree in Y of these polynomials. The inductive hypothesis gives $\deg q_{f_{g-1},t} < \deg f_{g-1}$ and $\deg P_i \leqslant \deg P_i f_{j_i} < \deg f_{g-1}$, for all i. On the other hand, the Abhyankar-Moh's relation $f = f_{g-1}^{n_g} - G$, where $G = a_{n_g-2} f_{g-1}^{n_g-2} + \cdots + a_0$ and $\deg a_i < \deg f_{g-1}$, gives the inequality $\deg G < (n_g - 1) \deg f_{g-1} = \deg f - \deg f_{g-1}$.

We also have $\deg[G, f_{j_i}] = \deg(G_X f_{j_i,Y} - G_Y f_{j_i,X}) \leqslant \deg G + \deg f_{j_i} - 1$. Therefore,

$$\begin{array}{rcl} \deg P_i[G, f_{j_i}] & \leqslant & \deg P_i + \deg G + \deg f_{j_i} - 1 \\ & < & \deg P_i f_{j_i} + \deg f - \deg f_{g-1} - 1 \\ & < & \deg f_{g-1} + \deg f - \deg f_{g-1} = \deg f, \end{array}$$

which together with the identity

$$\tilde{q}_{f_{g-1},t} = \sum_i P_i[f_{g-1}^{n_g} - G, f_{j_i}] = n_g f_{g-1}^{n_g-1} q_{f_{g-1},t} - \sum_i P_i[G, f_{j_i}],$$

give the estimate

$$\deg \tilde{q}_{f_{g-1},t} < \deg f, \quad \forall t \in S(f_{g-1})^*.$$

Claim 1: *For $t \in S(f_{g-1})^*$ we have $I(f, \tilde{q}_{f_{g-1},t}) = c(f) - 1 + n_g t$.*

Indeed, since no generator of $S(f)$ is multiple of p, we have from (3.2)

$$\begin{array}{rcl} I(f, P_i[f, f_{j_i}]) & = & I(f, P_i) + I(f, [f, f_{j_i}]) \\ & = & I(f, P_i) + I(f, f_{j_i}) + c(f) - 1 \\ & = & I(f, P_i f_{j_i}) + c(f) - 1. \end{array}$$

On the other hand, since the $P_i f_{j_i}$ are products of approximate roots of f_{g-1} (so, also of f), and $I(f_{g-1}, P_1 f_{j_1}) = t$, it follows that $I(f, P_1 f_{j_1}) = n_g t$. Now, since from (3.3), the intersection number $I(f, P_i f_{j_i})$ assumes its minimum value once for $i = 1$, when it is equal to t, we have

$$\begin{array}{rcl} I(f, \tilde{q}_{f_{g-1},t}) & = & I(f, \sum_i P_i[f, f_{j_i}]) \\ & = & I(f, P_1 f_{j_1}) + c(f) - 1 \\ & = & n_g t + c(f) - 1. \end{array}$$

■

The family of polynomials $\{\tilde{q}_{f_{g-1},t};\ t \in S(f_{g-1})^*\}$ just introduced will be used in the construction of the family $\{q_{f,s};\quad s \in S(f)^*\}$ as announced in the proposition.

To this end, observe that each element s of $S(f)^*$ decomposes uniquely as

$$s = n_g t + w v_g, \quad \text{with } t \in S(f_{g-1}),\ w \in \{0, 1, \dots, n_g - 1\}.$$

Now, we break up the analysis in three cases.

Case 1: $s = n_g t$. From Claim 1, we have

$$s + c(f) - 1 = n_g t + c(f) - 1 = I(f, \tilde{q}_{f_{g-1},t}).$$

The estimate on the degree of $\tilde{q}_{f_{g-1},t}$, made just before Claim 1, allows us to deduce that the series

$$q_{f,s} := \tilde{q}_{f_{g-1},t}$$

has all the required properties, which proves the proposition in this case.

Case 2: $s = v_g$. In this case $q_{f,v_g} := [f, f_{g-1}]$ works because, since $p \nmid v_g$, we have from Remark 3.3.2 that $I(f, q_{f,v_g}) = v_g + c(f) - 1$. Moreover, using the preceding notations and estimates we get

$$\begin{array}{rcl} \deg q_{f,v_g} & = & \deg\,[f_{g-1}^{n_g} - G, f_{g-1}] \\ & = & \deg\,[f_{g-1}, G] \\ & \leqslant & \deg G + \deg f_{g-1} - 1 \\ & < & (\deg f - \deg f_{g-1}) + \deg f_{g-1} - 1 \\ & < & \deg f. \end{array}$$

Case 3: $s > v_g$ and $w > 0$. Notice that from the conductor formula one gets that $c(f) - 1 = n_g(c(f_{g-1}) - 1) + (n_g - 1)v_g$, and since $n_i v_i < v_{i+1}$, it follows that $s > v_g > n_g(c(f_{g-1}) - 1)$. On the other hand, since $\gcd(v_g, n_g) = 1$, we have that $n_g \nmid s$.

The proposition, in this case, will be established by using the following result that gives a method to reduce degree while preserving intersection multiplicities with f and residual classes modulo $J(f)$.

Claim 2: *Let $s \in \mathbb{N}^*$ be such that $n_g \nmid s$ and $s > n_g(c(f_{g-1}) - 1)$. Suppose that we have a Y-polynomial h such that*

$$\deg h < \deg f \quad \textit{and} \quad I(f, h) = c(f) - 1 + s,$$

then there exists a Y-polynomial h', such that

$$\deg h' < \deg f - \deg f_{g-1}, \qquad I(f,h') = I(f,h),$$

and

$$h - h' = \sum_j \alpha_j \tilde{q}_{f_{g-1},u_j}, \quad \alpha_j \in k, \quad u_j \in S(f_{g-1}), \quad n_g u_j > s, \forall j.$$

Indeed we begin by dividing h by $f_{g-1}^{n_g-1}$. Then we get $h = f_{g-1}^{n_g-1}h_0'' + h_0'$ where $\deg h_0' < \deg f_{g-1}^{n_g-1} = \deg f - \deg f_{g-1}$. The rough idea of the proof is to eliminate the term $f_{g-1}^{n_g-1}h_0''$ in the preceding relation using the polynomials $\tilde{q}_{f_{g-1},u}$ where $u \in S(f_{g-1})^*$. This will be done iteratively, in possibly infinitely many steps, with the help of the following auxiliary result.

Claim 3: *With the same conditions as above, we have $I(f,h_0'') = n_g I(f_{g-1},h_0'')$ and $I(f,h_0'' f_{g-1}^{n_g-1}) \neq I(f,h_0')$.*

We will prove this claim after the conclusion of the proof of Claim 2, given below.

Using the formula $c(f) - 1 = n_g(c(f_{g-1}) - 1) + (n_g - 1)v_g$ and Claim 3, we get

$$I(f,h_0'' f_{g-1}^{n_g-1}) - (c(f) - 1) = n_g[I(f_{g-1},h_0'') - c(f_{g-1}) + 1].$$

On the other hand, since $I(f,h) - (c(f) - 1) = s$ and $n_g \nmid s$, it follows that

$$I(f,h_0') = I(f,h) < I(f, f_{g-1}^{n_g-1}h_0'').$$

So, from the first part of Claim 3 and the above inequality, we get

$$n_g I(f_{g-1},h_0'') = I(f,h_0'') > I(f,h) - I(f, f_{g-1}^{n_g-1}) = s + c(f) - 1 - (n_g - 1)v_g.$$

Defining $u_1 = I(f_{g-1},h_0'') - c(f_{g-1}) + 1$, it follows that

$$c(f_{g-1}) - 1 + u_1 = I(f_{g-1},h_0'') > \frac{s}{n_g} + c(f_{g-1}) - 1 > 2(c(f_{g-1}) - 1),$$

allowing us to conclude that $u_1 \in S(f_{g-1})^*$.

The inductive hypothesis guarantees the existence of a polynomial q_{f_{g-1},u_1} satisfying all requirements in (3.3) and the conclusion in Proposition 3.3.3.

From Claim 1, we have

$$I(f, \tilde{q}_{f_{g-1},u_1}) = c(f) - 1 + n_g u_1 = I(f, h_0'' f_{g-1}^{n_g-1}).$$

So, after multiplication by a suitable $\alpha_1 \in k^*$, we get that $h_1 = h_0'' f_{g-1}^{n_g-1} - \alpha_1 \tilde{q}_{f_{g-1},u_1}$ satisfies the inequality

$$I(f, h_1) > I(f, h_0'' f_{g-1}^{n_g-1}) > I(f, h) \; (= c(f) - 1 + s). \tag{3.4}$$

This allows us to write

$$h = h_1 + \alpha_1 \tilde{q}_{f_{g-1},u_1} + h_0', \quad \text{with} \quad I(f, h_1) > I(f, h) \quad \text{and} \quad I(f, h_0') = I(f, h).$$

From (3.4) we have that there exists $s_1 \in \mathbb{N}^*$ such that

$$I(f, h_1) = c(f) - 1 + s_1 > c(f) - 1 + n_g u_1 > c(f) - 1 + s.$$

So, $s_1 > s$ and $n_g u_1 > s$.

In the next step we proceed differently according to the divisibility of s_1 by n_g. Suppose $n_g \mid s_1$, say $s_1 = n_g u_2$. In this case, by the above inequality we have

$$2(c(f_{g-1}) - 1) < c(f_{g-1}) - 1 + u_1 < c(f_{g-1}) - 1 + u_2.$$

So, it follows that $u_2 \in S(f_{g-1})^*$. Hence, there exists a polynomial q_{f_{g-1},u_2} such that

$$I(f, h_1) = I(f, \tilde{q}_{f_{g-1},u_2})$$

and again we may choose $\alpha_2 \in k$ in such a way that if $h_2 = h_1 - \alpha_2 \tilde{q}_{f_{g-1},u_2}$, we have $I(f, h_2) > I(f, h_1)$. Hence, we get $h = h_2 + \alpha_1 \tilde{q}_{f_{g-1},u_1} + h_0' + \alpha_2 \tilde{q}_{f_{g-1},u_2} + h_1'$, where $h_1' = 0$, in this case. Notice that $n_g u_2 > n_g u_1 > s$.

If, however, $n_g \nmid s_1$, we are in position to repeat the preceding process of division by $f_{g-1}^{n_g-1}$ using, this time, h_1 instead of h. So $h_1 = f_{g-1}^{n_g-1} h_1'' + h_1'$. Again, we deduce that there exist $\alpha_2 \in k$ and $u_2 \in S(f_{g-1})^*$, with $n_g u_2 > s_1 > s$, such that if we define $h_2 = h_1'' f_{g-1}^{n_g-1} - \alpha_2 \tilde{q}_{f_{g-1},u_2}$, then we have

$$I(f, h_2) > I(f, h_1) > I(f, h).$$

So, by repeating this process we get

$$h = h_j + \sum_{i=1}^{j} \alpha_i \tilde{q}_{f_{g-1},u_i} + \sum_{i=0}^{j-1} h'_i,$$

with $I(f, h'_i) < I(f, h'_{i+1})$, if $h'_i \neq 0$, and $I(f, h_i) < I(f, h_{i+1})$. Since all power series appearing in the above sum have degree less than $\deg f$, it follows, in view of Proposition 3.3.1, that $h_j \to 0$ in the $\mathfrak{m}$-adic topology of $\mathcal{R}$ and the family $\{h'_i\}_{i\in\mathbb{N}}$ is summable. Taking $h' = \sum_j h'_j$ we get Claim 2. ■

Finally it remains to prove Claim 3. If f is any irreducible Weierstrass polynomial of degree n, then it is easy to see from Proposition 3.3.1 that the set V_{n-1} of all polynomials in Y of degree less than n with coefficients in $k[[X]]$ is a free $k[[X]]$-module with basis

$$\left\{ f^J = f_0^{j_0} f_1^{j_1} \cdots f_{g-1}^{j_{g-1}};\ J = (j_0, \ldots, j_{g-1}),\ 0 \leqslant j_i < n_{i+1},\ i = 0, \ldots, g-1 \right\}.$$

So, every element $h \in V_{n-1}$ may be written uniquely as

$$h = \sum_J a_J(X) f^J = f_{g-1}^{n_g - 1} h'' + h', \qquad a_J(X) \in k[[X]],$$

where

$$h'' = \sum_{j_{g-1} = n_g - 1} a_J(X) f_0^{j_0} \cdots f_{g-2}^{j_{g-2}}, \qquad h' = \sum_{j_{g-1} \leqslant n_g - 2} a_J(X) f_0^{j_0} \cdots f_{g-1}^{j_{g-1}}.$$

First of all we will check that $I(f, h') \neq I(f, f_{g-1}^{n_g-1} h'')$. In fact, in h' there is a unique term such that

$$I(f, h') = I(f, a_J(X) f_0^{j_0} \cdots f_{g-1}^{j_{g-1}}) = \sum_{i=-1}^{g-1} j_i v_{i+1},$$

where $j_{-1} = \mathrm{ord}_X a_J(X)$. Also, in $f_{g-1}^{n_g-1} h''$ there is a unique term satisfying

$$I(f, f_{g-1}^{n_g-1} h'') = I(f, a_L(X) f_0^{l_0} \cdots f_{g-2}^{l_{g-2}} f_{g-1}^{n_g-1}) = \sum_{i=-1}^{g-2} l_i v_{i+1} + (n_g - 1) v_g,$$

where $l_{-1} = \mathrm{ord}_X(a_L(X))$.

Since each element in $S(f)$ is written in a unique way as $\sum_{i=-1}^{g-1} j_i v_i$ with $j_{-1} \in \mathbb{N}$

and $0 \leqslant j_i \leqslant n_{i+1} - 1$, the inequality follows.

Also, it is clear from the way we wrote h'' that $I(f, h'') = n_g I(f_{g-1}, h'') \in n_g S(f_{g-1})$.

Now, to finish the proof of Claim 3 we only need to check that h'' and h' are indeed the quotient and the remainder, respectively, of the division of h by $f_{g-1}^{n_g-1}$. We will do this by estimating the degree of h' and , hence, conclude by the uniqueness of the remainder and the quotient in the euclidean algorithm. Indeed, for every summand in h' we have $\deg\,(a_J(X) f_0^{j_0} \cdots f_{g-1}^{j_{g-1}}) < \deg f_{g-1}^{n_g-1}$, which shows that $\deg h' < \deg f_{g-1}^{n_g-1}$. ■

Now we return to the construction of the polynomial $q_{f,s}$ in the remaining Case 3, that is, when $s = n_g t + v_g w$ with $s > v_g$ and $w > 0$.

Observe that if $t = 0$ and $w = 1$, then from Case 2 we have $q_{f,v_g} = [f, f_{g-1}]$. Now, we apply Claim 2 to $h = q_{f,v_g}$ in order to find $h' = (q_{f,v_g})'$ with degree less than $\deg f - \deg f_{g-1}$ satisfying

$$I(f, (q_{f,v_g})') = I(f, q_{v_g,f}) = c(f) - 1 + v_g$$

and

$$(q_{f,v_g})' = q_{f,v_g} + \sum_j \alpha_j \tilde{q}_{f_{g-1},u_j}.$$

Using this, we define $q_{f,2v_g} := f_{g-1}(q_{f,v_g})'$. Clearly, we have $\deg q_{f,2v_g} < \deg f$ and $I(f, q_{f,2v_g}) = c(f) - 1 + 2v_g$. Hence, it remains to show that $q_{f,2v_g}$ satisfies (3.3) in order to make possible our inductive process.

We have

$$q_{f,2v_g} = f_{g-1}[f, f_{g-1}] + \sum_j \alpha_j f_{g-1} \tilde{q}_{f_{g-1},u_j} = \sum_i P_i [f, f_{j_i}].$$

This shows that $q_{f,2v_g}$ has the required format. Finally, we need to check the statement about intersection indices. We are going to show that $P_1 = f_{j_1} = f_{g-1}$. In order to do so, it is enough to show that for each index j in the above sum, the polynomial

$$\tilde{q}_{f_{g-1},u_j} = \sum_l P'_l [f, f_{j_l}],$$

where f_{j_l} is one of the approximate roots $f_{-1}, f_0, \ldots, f_{g-2}$ and the P'_l are monomials in these approximate roots, is such that $I(f, f_{g-1} P'_i f_{j_i}) > 2v_g$. Indeed, from the inductive hypothesis we have

$$I(f, f_{g-1}P_l' f_{j_l}) = v_g + I(f, P_l' f_{j_l}) = v_g + n_g I(f_{g-1}, P_l' f_{j_l}) > v_g + n_g u_1 > 2v_g,$$

where the last strict inequality is justified by the fact that from Claim 2 one has $n_g u_1 > v_g$.

We apply again Claim 2 to obtain $(q_{f,2v_g})'$ which multiplied by f_{g-1} produces $q_{f,3v_g}$. Now, we repeat this procedure until we get the polynomial $q_{f,wv_g} = f_{g-1}(q_{f,(w-1)v_g})'$, satisfying the proposition for $wv_g \in S(f)^*$. Since $s = n_g t + wv_g$, we consider the polynomial $q_{f_{g-1},t} = \sum P_i''[f_{g-1}, f_{m_i}]$ and collect P_1'', f_{m_1} so that $I(f_{g-1}, P_1'' f_{m_1}) = t$. Finally, define

$$q_{f,s} := P_1'' f_{m_1}(q_{f,wv_g})'.$$

It is now immediate to verify that $q_{f,s}$ satisfies (3.3) and the conclusion of the proposition, finishing its proof. ■

With these tools at hands, we may conclude the proof of Theorem 3.2.5.

Proof of the Theorem 3.2.5: Choose $\mathcal{F}$ with minimal number of elements, so from Lemma 3.2.8 it follows that $\#\mathcal{F} = c(f)$. We will show that the set $\overline{\mathcal{F}}$ generates $\mathcal{R}/J(f)$ as a k-vector space. In particular, this will show also that $\mu(f) \leqslant c(f)$ when $S(f)$ is tame. In order to do this it is enough to show that there exists a decomposition $\mathcal{R} = \langle \mathcal{F} \rangle + J(f)$, where $\langle \mathcal{F} \rangle$ denotes the k-vector space spanned by the elements of $\mathcal{F} = \{\varphi_1, \ldots, \varphi_{c(f)}\}$.

Given any element $h \in \mathcal{R}$ we can divide it by the partial derivative f_Y which, under our assumptions, is a Y-polynomial of degree $v_0 - 1$. The remainder of the division is a Y-polynomial h' of degree less than $v_0 - 1$ and it is sufficient to show that h' belongs to $\langle \mathcal{F} \rangle + J(f)$.

If $I(f, h') \in S(f) \setminus (S(f) + c(f) - 1)$ then, according to the definition of $\mathcal{F}$, there is an element $\varphi_{i_{s_0}}$ such that $s_0 := I(f, h') = I(f, \varphi_{i_{s_0}})$. Hence, there is a constant $\alpha_{s_0} \in k$ such that

$$I(f, h' - \alpha_{s_0}\varphi_{i_{s_0}}) =: s_1 > s_0.$$

If, otherwise $I(f, h') = s_0 = s_0' + c(f) - 1 \in S(f)^* + c(f) - 1$, then choose an element $q_{f,s_0'}$ in $J(f)$ polynomial in Y of degree less then $\deg f$, such that $s_0 = I(f, q_{f,s_0'})$. Hence, there is a constant $\beta_{s_0} \in k$ such that

$$I(f, h' - \beta_{s_0} q_{f,s_0'}) =: s_1 > s_0.$$

We carry on this process that increases intersection indices to eventually achieve

$$s_r = I(f, h' - \sum_s \beta_s q_{f,s'} - \sum_s \alpha_s \varphi_{i_s}) \in S(f)^* + c(f) - 1, \quad \forall r \geqslant N.$$

Since the elements in $S(f)^* + c(f) - 1$ may be realized as intersections indices of f with elements in $J(f) \cap V_{n-1}$ (cf. Proposition 3.3.3), we produce an element

$$h' - \sum_s \alpha_s \varphi_{i_s} - \sum_j \beta_s q_{f,s'}$$

whose intersection multiplicity with f is big enough and whose degree is less than $\deg f$, hence from Proposition 3.3.1 it belongs to the Jacobian ideal $J(f)$. ∎

The classical Milnor's Formula for plane curves singularities states that if $f = f_1 \cdots f_r \in \mathbb{C}[[X,Y]]$ is a possibly many branched and reduced power series over $\mathbb{C}$ then $\mu(f) = 2\delta(f) + 1 - r$. A first possible tentative to extend the result obtained in Theorem 3.2.4 is:

If all branches $f_1, \cdots, f_r$ have tame semigroups then the preceding formula continues to hold.

We include here some examples to show that this is NOT true.

Let $f = (Y^2 - X^3)^2 - X^{11}Y$ and $g = (Y^2 - X^3 + X^2Y)^2 - X^{11}Y$. Then $I(f,g) = 28$ and $S(f) = S(g) = \langle 4, 6, 25 \rangle$. Here one can compute

$$\delta(fg) = \delta(f) + \delta(g) + I(f,g) = 14 + 14 + 28 = 56.$$

Hence $2\delta(fg) + 1 - r = 111$. The behaviour of the reduced 2-branched plane curve singularity defined by the equation fg with respect to μ-stability and the value of $e_0(T(fg)) = \mu(\mathcal{O}_{fg})$ according to the characteristic p of the ground field is described below. All computations are performed using the software *Singular*, [DGPS]. See the Appendix A for explanations about the procedures.

Example 3.3.4. ($p = 7$) *Here $(fg)^4 \in \mathfrak{m}T(fg)^4$ so that fg is μ-stable. Computation shows that $\mu(\mathcal{O}_{fg}) = \mu(fg) = 112 > 111$ so that Milnor's Formula does not hold. Notice that the semigroups of the branches are tame in characteristic $p = 7$, but $p \mid I(f,g)$.*

Example 3.3.5. ($p = 5$) *Here* $(fg)^3 \in \mathfrak{m}T(fg)^3$ *so that again* fg *is* μ*-stable. Computation now shows that* $\mu(\mathcal{O}_{fg}) = \mu(fg) = 111$ *and Milnor's Formula does hold. Notice that the semigroups of the branches are wild in characteristic* $p = 5$.

Example 3.3.6. ($p = 13$) *Here computation shows that* $\mu(fg) = 124$ *and* $\mu((1 + X)fg) = 114$ *so that* fg *is not* μ*-stable. Computation also suggests that* $\mu(\mathcal{O}_{fg}) = \mu(fg) = 114$ *and Milnor's Formula does not hold. Notice that the semigroups of the branches are tame in characteristic* $p = 13$ *and* $p \nmid I(f, g)$.

CHAPTER 4

Differentials

4.1 The module of differentials of a plane curve

Let $\mathcal{O}$ be an algebroid singular plane curve over an algebraically closed field k of characteristic $p \geq 0$. Consider the $\mathcal{O}$-module $\Omega = \Omega_k(\mathcal{O})$ of Kähler differentials of $\mathcal{O}$ over k and let $d\colon \mathcal{O} \to \Omega_k(\mathcal{O})$ be the universal derivation. Choose an embedding of $\mathcal{O}$ by means of an epimorphism $\varphi\colon \mathcal{R} \to \mathcal{O}$, where $\mathcal{R} = k[[X,Y]]$, whose kernel is an ideal $\mathfrak{A} = \langle f \rangle$. This epimorphism induces the conormal exact sequence

$$\mathfrak{A}/\mathfrak{A}^2 \to \mathcal{O} \otimes_k \Omega_k(\mathcal{R}) \to \Omega \to 0,$$

which gives us an isomorphism

$$\Omega \simeq \frac{\mathcal{O}dX \oplus \mathcal{O}dY}{(f_X dX + f_Y dY)\mathcal{O}}.$$

For $h \in \mathcal{R}$, we will write h_x for $h_X(x,y) \in \mathcal{O}$ and similarly for h_y. In view of the relation $f_x dx + f_y dy = 0$, for any differential $\omega = adx + bdy \in \Omega$ we have

$$f_y\omega = f_y(adx + bdy) = (af_y - bf_x)dx,$$

so we have an $\mathcal{O}$-modules isomorphism

$$f_y\Omega \simeq (\mathcal{O}f_x + \mathcal{O}f_y)dx.$$

A differential $\omega \in \Omega$ will be called an *exact differential*, if there exists $h \in \mathcal{O}$ such that $\omega = dh = h_x dx + h_y dy$. These differentials form a k-vector subspace of Ω that will be denoted by $d\mathcal{O}$.

A differential ω will be called a *torsion differential* if there exists a nonzero divisor $\xi \in \mathcal{O}$ such that $\xi\omega = 0$. These differentials form an $\mathcal{O}$-submodule of Ω denoted by $\mathcal{T}$.

Since $\mathcal{O}$ is noetherian and Ω is a finitely generated $\mathcal{O}$-module, it follows that Ω is a noetherian module, hence $\mathcal{T}$ is a finitely generated submodule ([AM] pp. 75,76). Now, since $\mathcal{T}$ is finitely generated and all its elements are torsion elements, it follows that the annihilator of $\mathcal{T}$ contains a nonzero divisor ξ.

We now show that the $\mathcal{O}$-module $\mathcal{T}$ has finite length. For this, it is enough to show that all the prime ideals that contain the annihilator of $\mathcal{T}$ are maximal (cf. [Eisenbud], Corollary 2.17). In fact, since $\dim \mathcal{O} = 1$ and the annihilator of $\mathcal{T}$ contains a nonzero divisor ξ, it follows that ξ does not belong to any of the minimal primes of (0), so the only prime that contains ξ is the maximal ideal $\mathfrak{m}$.

In the sequel we will prove in greater generality a result that is found in [Za2] in the zero characteristic and irreducible context that relates the length $\ell_{\mathcal{O}}(\mathcal{T})$ of the torsion submodule $\mathcal{T}$ as an $\mathcal{O}$-module and the Tjurina number $\tau(\mathcal{O})$. Before, we will need a couple of lemmas.

Lemma 4.1.1. *Let $g \in \mathcal{R}$ without multiple factors and let h be an irreducible factor of g. Then the set $\{(c:d) \in \mathbb{P}^1_k;\ h \mid cg_X + dg_Y\}$ is finite.*

Proof: Suppose that $g = hg'$ with $h, g' \in \mathcal{R}$ coprime. We then have

$$cg_X + dg_Y = h(cg'_X + dg'_Y) + g'(ch_X + dh_Y).$$

So,

$$h \mid cg_X + dg_Y \iff h \mid g'(ch_X + dh_Y) \iff I(h, ch_X + dh_Y) = \infty.$$

We now analyze this last intersection multiplicity.

If $I_X = I(h, h_X) \neq I(h, h_Y) = I_Y$, then for general $(c:d) \in \mathbb{P}^1_k$, one has

$$I(h, ch_X + dh_Y) = \min\{I(h, h_X), I(h, h_Y)\} < \infty.$$

If $I_X = I_Y = I$, then $I \neq \infty$, because otherwise, we would have $h \mid h_X$ and $h \mid h_Y$, hence $h = 0$ or h is not reduced, which is a contradiction (cf. [Bou]).

So, since $I \neq \infty$, there is a unique $(c:d) \in \mathbb{P}^1_k$ such that

$I(h, ch_X + dh_Y) > I$, hence for a general element $(c:d)$ in $\mathbb{P}^1_k$ one has $I(h, ch_X + dh_Y) = I < \infty$.

■

Lemma 4.1.2. *Let $g \in \mathcal{R}$ be reduced. Then for a general linear automorphism $\varphi(X,Y) = (aX+bY, cX+dY)$ of $\mathcal{R}$ one has that $f = g \circ \varphi$ is such that $I(f, f_Y) < \infty$.*

Proof: Let f_1 be an irreducible component of f, then $f_1 = h \circ \varphi$ for some irreducible component h of g. We have that

$$f_1 \mid f_Y \iff h \circ \varphi \mid cg_X \circ \varphi + dg_Y \circ \varphi.$$

Since $cg_X \circ \varphi + dg_Y \circ \varphi = (cg_X + dg_Y) \circ \varphi$, it follows that

$$f_1 \nmid f_Y \iff h \nmid cg_X + dg_Y.$$

From Lemma 4.1.1 this occurs for a general point $(c:d)$ in $\mathbb{P}^1_k$. So, for a general φ no irreducible factor of f will divide f_Y.

■

Theorem 4.1.3. *If $\mathcal{O}$ is an algebroid plane curve with an isolated singularity, then $\ell_{\mathcal{O}}(\mathcal{T}) = \tau(\mathcal{O})$.*

Proof: (cf. [Za2]) From Lemma 4.1.2 we may choose an equation f for $\mathcal{O}$ such that $I(f, f_Y) \neq \infty$, so the image f_y of f_Y in $\mathcal{O}$ is not a zero divisor.

Consider the ideal $\mathfrak{B} = \{B \in \mathcal{O} \,|\, B\, f_x \in f_y\, \mathcal{O}\}$ of $\mathcal{O}$ and let $\varphi : \mathfrak{B} \to \Omega$ the homomorphism of $\mathcal{O}$-modules defined by $B \mapsto B\frac{f_x}{f_y}\, dx + B\, dy$. First, we are going to show that

(i) $\varphi(\mathfrak{B}) = \mathcal{T}$ and

(ii) $\ker \varphi = \mathcal{O}\, f_y$.

Indeed, let $\omega = B\frac{f_x}{f_y}\, dx + B\, dy \in \varphi(\mathfrak{B})$. We have that

$$f_y\, \omega = B\, f_x\, dx + B\, f_y\, dy = B\,(f_x\, dx + f_y\, dy) = 0.$$

Hence $\omega \in \mathcal{T}$, because f_y is not a zero divisor in $\mathcal{O}$.

Conversely, if $\omega = a\,dx + b\,dy \in \mathcal{T}$, we have $0 = \xi\,\omega = \xi\,a\,dx + \xi\,b\,dy$, for some nonzero divisor ξ. Hence there is some $\eta \in \mathcal{O}$ such that $\xi\,a\,dx + \xi\,b\,dy = \eta\,(f_x\,dx + f_y\,dy)$, so that $\xi\,a = \eta\,f_x$ and $\xi\,b = \eta\,f_y$. This implies that $\xi\,a\,f_y = \xi\,b\,f_x$, hence $a\,f_y = b\,f_x$, so $a \in \mathfrak{B}$ and $\varphi(a) = \omega$. This proves *(i)*. The verification of *(ii)* follows the same lines.

Hence we have the following $\mathcal{O}$-modules isomorphism:

$$\mathcal{T} \simeq \mathfrak{B}/\mathcal{O}\,f_y. \tag{4.1}$$

Consider the following sequence of natural $\mathcal{O}$-epimorphisms

$$\mathcal{O} \to \mathcal{O}\,f_x \to \mathcal{O}\,f_x/(\mathcal{O}\,f_x \cap \mathcal{O}\,f_y) \simeq (\mathcal{O}\,f_x + \mathcal{O}\,f_y)/\mathcal{O}\,f_y = \mathrm{j}/\mathcal{O}\,f_y$$

and let ψ be the composite map $\mathcal{O} \to \mathrm{j}/\mathcal{O}\,f_y \to 0$.

We have

$$\xi \in \ker\psi \;\Leftrightarrow\; f_x\,\xi \in \mathcal{O}\,f_y \;\Leftrightarrow\; \xi \in \mathfrak{B}$$

and, therefore,

$$\mathcal{O}/\mathfrak{B} \simeq \mathrm{j}/\mathcal{O}\,f_y. \tag{4.2}$$

Since $\mathcal{O}\,f_y \subset \mathrm{j} \subset \mathcal{O}$, $\mathcal{O}\,f_y \subset \mathfrak{A} \subset \mathcal{O}$ and $\ell_{\mathcal{O}}\Big(\mathcal{O}/\mathcal{O}\,f_y\Big) = I(f, f_Y) < \infty$, we have

$$\ell_{\mathcal{O}}\Big(\mathcal{O}/\mathcal{O}\,f_y\Big) = \ell_{\mathcal{O}}\Big(\mathcal{O}/\mathrm{j}\Big) + \ell_{\mathcal{O}}\Big(\mathrm{j}/\mathcal{O}\,f_y\Big)$$

and

$$\ell_{\mathcal{O}}\Big(\mathcal{O}/\mathcal{O}\,f_y\Big) = \ell_{\mathcal{O}}\Big(\mathcal{O}/\mathfrak{B}\Big) + \ell_{\mathcal{O}}\Big(\mathfrak{B}/\mathcal{O}\,f_y\Big).$$

This gives, in view of (4.1) and (4.2), that

$$\ell_{\mathcal{O}}\Big(\mathcal{T}\Big) = \ell_{\mathcal{O}}\Big(\mathfrak{B}/\mathcal{O}\,f_y\Big) = \ell_{\mathcal{O}}\Big(\mathcal{O}/\mathrm{j}\Big) = \tau(\mathcal{O}).$$

■

Example 4.1.4. *Let* $f = Y^p - X^{p+1} \in k[[X,Y]]$, *where* $p = \mathrm{char} k$ *and define* $\mathcal{O} = \mathcal{O}_f$. *Since* $0 = f_x dx + f_y dy = -x^p dx$, *it follows that* $x^i y^j dx$ *with* $0 \le i, j \le p-1$ *are linearly* *independent torsion differentials. Since $\ell(\mathcal{T}) = \tau(\mathcal{O}) = p^2$, it follows that these differentials form a basis for $\mathcal{T}$. Observe that the differentials dx^i, for $i = 0, \ldots, p-1$ are exact torsion differentials, which is a phenomenon that only occurs in positive characteristic since if $p = 0$ there are no other exact torsion differentials than* 0.

Notice that when if $\mathcal{O}$ is irreducible and when the semigroup $S(\mathcal{O})$ is tame, one has that $\mu(\mathcal{O}) = c(\mathcal{O}) = c$, hence

$$\ell_{\mathcal{O}}(\mathcal{T}) = \tau(\mathcal{O}) \leqslant \mu(\mathcal{O}) = c.$$

This may fail when the semigroup of $\mathcal{O}$ is not tame, as shows the following example:

Example 4.1.5. *Suppose that the characteristic of k is $p = 3$ and consider the plane branch $\mathcal{O}$ defined by the equation $f = Y^3 - X^{11}$. The semigroup of $\mathcal{O}$ is $\langle 3, 11\rangle$, hence it is not tame and its conductor c is* 20. *An easy computation shows that* $\tau(\mathcal{O}) = 30$, *so, in this example, one has $\tau(\mathcal{O}) > c$.*

4.2 Values of differentials of branches

Let $\mathcal{O}$ be a plane branch with an equation f. Given an embedding $\psi : \mathcal{O} \hookrightarrow \overline{\mathcal{O}} \simeq k[[t]]$, we obtain a parametrization $x = x(t)$, $y = y(t)$ for f. We assume from now on that this parametrization is primitive, that is, the field of quotients of $\psi(\mathcal{O})$ coincides with that of $k[[t]]$. From $f(x(t), y(t)) = 0$ we get

$$f_x\, x' + f_y\, y' = 0 \tag{4.3}$$

where $x' = \frac{dx}{dt}$ and similarly for y'.

In terms of differentials, the embedding ψ induces a natural map (denoted with the same symbol) $\psi : \Omega \longrightarrow \Omega_k(\overline{\mathcal{O}}) \simeq k[[t]]\, dt$, given by

$$\psi(a\, dx + b\, dy) = (a(x(t), y(t))\, x'(t) + b(x(t), y(t))\, y'(t))\, dt$$

For simplicity we will use the notation $\Omega_k(\overline{\mathcal{O}}) = \overline{\Omega}$. The above $\mathcal{O}$-module homomorphism ψ induces on $\overline{\Omega}$ the structure of an $\mathcal{O}$-module and its kernel is exactly $\mathcal{T}$. Hence it induces an exact sequence

$$0 \to \mathcal{T} \to \Omega \to \psi(\Omega) \to 0.$$

We may characterize the exact torsion differentials as follows. If $h \in \mathcal{O}$ is such that $d\,h \in \mathcal{T}$, then $0 = \psi(d\,h)$ and this in turn implies that $h_x(x,y)\,x' + h_y(x,y)\,y' = \frac{dh}{dt} = 0$. So, we have

$$\begin{aligned} d\,\mathcal{O} \cap \mathcal{T} &= \{d\,h \mid h \in k\} = \{0\}, \text{ if } p = 0 \text{ and} \\ d\,\mathcal{O} \cap \mathcal{T} &= \{d\,h \mid h \in \mathcal{O} \cap \overline{\mathcal{O}}^p\}, \text{ if } p > 0. \end{aligned} \tag{4.4}$$

Example 4.2.1. *Consider $f = Y^4 - X^9$ and suppose that $p = 13$. Then $\mathcal{O} = \mathcal{O}_f$ has a parametrization $x = t^4$, $y = t^9$. Consider the element xy. We have that $d(xy)$ is exact and belongs to $\mathcal{T} \setminus \{0\}$. Indeed, we have that $\psi(xy) = t^{13}$, so $\psi(d(xy)) = \frac{dt^{13}}{dt} = 0$, hence $d(xy) \in \mathcal{T}$. This example shows that even if the semigroup of $\mathcal{O}$ is tame, one may have $d\,\mathcal{O} \cap \mathcal{T} \neq \{0\}$.*

Example 4.2.2. *Suppose $\operatorname{char} k = p > 0$ and let $f = Y^p - X^{p+1} \in \mathcal{R}$. In $\mathcal{O} = \mathcal{R}/\langle f \rangle$ we have the relation $y^p = x^{p+1}$ so that*

$$\mathcal{O} = k[[x]] \oplus k[[x]]\,y \oplus \cdots \oplus k[[x]]\,y^{p-1}$$

and we see that any element of $\mathcal{O}$ is a k-linear combination of elements of the form $x^i\,y^j$ with $0 \leqslant j \leqslant p-1$ and $i \in \mathbb{N}$. Parametrizing $\mathcal{O}$ with $x = t^p$, $y = t^{p+1}$ we see that such a monomial lies in $\overline{\mathcal{O}}^p = k[[t]]^p = k[[x]]$ if and only if $j = 0$. It follows that $0, d\,x, \ldots, d\,x^{p-1}$ are distinct exact torsion differentials.

We will define the *value* of a non-torsion differential as follows:

If $\omega = adx + bdy \in \Omega$, we set

$$v(\omega) = \operatorname{ord}_t(a(x(t),y(t))\,x'(t) + b(x(t),y(t))\,y'(t)) + 1,$$

and then define

$$\Lambda(\mathcal{O}) = \{v(\omega);\ \omega \in \Omega \setminus \mathcal{T}\}.$$

This set is called the set of values of differentials.

If $p = 0$, it is known that $S(\mathcal{O})^* \subseteq \Lambda(\mathcal{O})$. Indeed, in this case one has

$$S(\mathcal{O})^* = v(d\mathcal{O} \setminus \{0\}) \subset \Lambda(\mathcal{O}),$$

since for all $h \in \mathfrak{m}$

$$v(dh) = \mathrm{ord}_t h'(x(t), y(t)) + 1 = \mathrm{ord}_t h(x(t), y(t)) - 1 + 1 = v(h).$$

If $p > 0$, even in the tame case, the equality $S(\mathcal{O})^* = v(d\mathcal{O} \setminus \{0\})$ may fail as shows the following example.

Example 4.2.3. *Let $p = 11$ and consider $f = Y^4 - X^7$. Put $\mathcal{O} = \mathcal{O}_f$. Here $S(\mathcal{O}) = \langle 4, 7\rangle$ is tame, but, as it is easy to see, $11 = 4 + 7 \in S(\mathcal{O})^*$ is not the value of any* exact *differential.*

So, in this case, $S(\mathcal{O})^* \not\subset v(d\mathcal{O} \setminus \{0\})$.

One can easily check that, in general,

$$S(\mathcal{O}) \setminus p\mathbb{N} \subset v(d\mathcal{O} \setminus \{0\}),$$

but the other inclusion, $v(d\mathcal{O} \setminus \{0\}) \subset S(\mathcal{O})^*$ may fail, as well as the inclusion $S(\mathcal{O})^* \subset \Lambda(\mathcal{O})$, as will be shown in Example 4.2.4 below.

Let $\omega = a\,dx + b\,dy$, then

$$\begin{aligned} \psi(f_y\omega) &= f_y(x(t), y(t)) \cdot \Big(a(x(t), y(t))\, x'(t) + b(x(t), y(t))\, y'(t)\Big)\, dt \\ &= \psi(a\, f_y - b\, f_x)\, x'\, dt. \end{aligned}$$

Since $v(f_y) = c + v(x')$ (see Theorem 3.1.1) and $v(a\, f_y - b\, f_x) \in \nu(J(f)^*) = \nu(T(f)^*)$, it follows that

$$\nu(T(f)^*) = \nu(J(f)^*) = \Lambda(\mathcal{O}) + c - 1.$$

Example 4.2.4. *Let $p = 7$ and consider $\mathcal{O} = \mathcal{O}_f$, where $f = Y^3 + X^7 + X^5Y \in \mathcal{R}$. We have that $S(\mathcal{O}) = \langle 3, 7\rangle$, which is not tame and $c = 12$. So, $v([f, f_{-1}]) = 14 = v(d\,x) + 11$ and $v([f, f_0]) = 19 = v(d\,y) + 11$. Hence $v(d\,x) = 3$ and $v(d\,y) = 8 \in v(d\,\mathcal{O}) \setminus S(\mathcal{O})$. This shows that the inclusion $v(d\mathcal{O} \setminus \{0\}) \subset S(\mathcal{O})$ may fail.*

In this example the inclusion $S(\mathcal{O})^* \subset \Lambda(\mathcal{O})$ also fails. Indeed, we have that $7 \in S(\mathcal{O}) \setminus \Lambda(\mathcal{O})$ because if $7 \in \Lambda(\mathcal{O})$ there might exist a differential form $\omega = a\,d\,x + b\,d\,y$ such that $7 = v(\omega) = v(a\, f_y - b\, f_x) - 11$. Hence $18 = v(a\, f_y - b\, f_x) \geqslant \min\{v(a) + 14,\ v(b) + 19\}$.

The only possibility is that $18 = v(a) + 14$*, so* $v(a) = 4$*. This is impossible because* $4 \notin S(\mathcal{O})$*.*

However, it is remarkable that in the tame case we still have $S(\mathcal{O})^* \subseteq \Lambda(\mathcal{O})$, not because $S(\mathcal{O})^* = v(d(\mathcal{O} \setminus \{0\}))$, but for a completely different reason.

Indeed, in general, if $S(\mathcal{O}) = \langle v_0, \dots, v_g \rangle$, then

$$S(\mathcal{O})^* = \bigcup_{i=-1}^{g-1} \Big(v_{i+1} + S(\mathcal{O}) \Big).$$

Hence if $S(\mathcal{O})$ is *tame* and if we choose elements $h_{-1}, h_0, \dots, h_{g-1} \in \mathcal{O}$ such that $v(h_i) = v_{i+1}$, then Remark 3.3.2 implies that

$$v([f, h_i]) = c + v(h_i') = v_{i+1} - 1 + c = v(d\, f_i) + c - 1.$$

This gives

$$S(\mathcal{O})^* + c - 1 \subseteq \Lambda(\mathcal{O}) + c - 1,$$

so we still conclude *in the tame case* that $S(\mathcal{O})^* \subseteq \Lambda(\mathcal{O})$.

In the next section we will discuss in characteristic $p > 0$ some properties of plane branches known to hold in characteristic zero. To do this, we will need a result due to Berger in [Be] whose proof we borrowed from Assi and Sathaye ([A-S]).

Let us recall the differential operator defined on $\mathcal{O}$ by

$$D_f(g) = [f, g] = f_x g_y - f_y g_x,$$

where $f \in \mathcal{R}$ is irreducible. D_f may be extended in a natural way into a differential operator on the field of fractions $\mathcal{K}$ of $\mathcal{O}$ in the following way:

$$D_f\left(\frac{g}{h}\right) = \left[f, \frac{g}{h}\right] := \frac{hD_f(g) - gD_f(h)}{h^2} = \frac{h[f,g] - g[f,h]}{h^2}.$$

We have $D_f(\overline{\mathcal{O}}) \subset \overline{\mathcal{O}}$, where $\overline{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in $\mathcal{K} \simeq k((t))$. Indeed, $\frac{g}{h} \in \overline{\mathcal{O}}$ if and only if $v(g) \geqslant v(h)$. Then, using Corollary 3.1.6,

$$\begin{aligned} v\left(D_f\left(\frac{g}{h}\right)\right) &\geqslant \min\{v([f,g])-v(h), v([f,h])+v(g)-2v(h)\} \\ &= -v(h)+\min\{v([f,g]), v([f,h])+v\left(\frac{g}{h}\right)\} \\ &\geqslant -v(h)+\min\{v(g)+c-1, v(g)+c-1\} \\ &= v\left(\frac{g}{h}\right)+c-1 \geqslant c-1 \geqslant 0 \end{aligned}$$

so that $D_f(\overline{\mathcal{O}}) \subset \overline{\mathcal{O}}$.

Using the relation $f_x dx + f_y dy = 0$, one can easily show that for any $u \in \mathcal{K}$ one has

$$f_y du = -[f,u]dx \text{ and } f_x du = -[f,u]dy \tag{4.5}$$

Hence, since x or y is separable, it follows that $[f,u]=0$ if and only if $u'=0$, that is $u \in k((t^p))$. This also shows that if $u \notin k((t^p))$, then the differential $\omega = \frac{du}{[f,u]} \in \Omega_k(\mathcal{K})$ is independent of u and is equal to either $-\frac{dx}{f_y}$ or $\frac{dy}{f_x}$ and these two differentials coincide when both are defined.

Let $u \in \mathcal{K}$ is such that the differential $\omega = \frac{du}{[f,u]}$ is well defined then if we compute the values in both sides of (4.5), we get that $v([f,u]) = c + v(u')$ which is the same as $v(\omega) = -c$.

Recall that we defined the conductor ideal of $\overline{\mathcal{O}}$ in $\mathcal{O}$ as being

$$\mathcal{C}(\mathcal{O}) = \{h \in \overline{\mathcal{O}};\ h\overline{\mathcal{O}} \subset \mathcal{O}\}.$$

Let us define the ideal I^* of $\overline{\mathcal{O}}$ as the ideal generated by all the elements $[f,u]$, where u sweeps $\overline{\mathcal{O}}$, that is,

$$I^* = \{[f,u];\ u \in \overline{\mathcal{O}}\}\,\overline{\mathcal{O}}.$$

If we denote by j the image of $J(f)$ or of $T(f)$ in $\mathcal{O}$, then we have the following result:

Lemma 4.2.5. *One has that*

i) $I^* = \mathcal{C}(\mathcal{O})$*;*

ii) The multiplication map $m_\omega \colon \mathcal{O} \to \Omega_k(\mathcal{K})$, $h \mapsto \psi(h)\omega$ *is an injective homomorphism of* $\mathcal{O}$*-modules;*

iii) $m_\omega(\mathcal{C}(\mathcal{O})) = \overline{\Omega}$;

iv) $m_\omega(\mathrm{j}) = \psi(\Omega)$.

Proof: i) Since $v([f,t]) = c + v(t')$, then $v([f,t]) = c$, which implies easily the result.
ii) This is obvious because $\psi\colon \mathcal{O} \to \overline{\mathcal{O}}$ is injective and $\Omega_k(\mathcal{K}) \simeq k((t))dt$ is free.
iii) Since $v(\omega) = -c$,
and for all $h \in \mathcal{O}$, one has $v(h) \geq c$ if and only if $h \in \mathcal{C}(\mathcal{O})$, and since $0 \neq \rho \in \overline{\Omega} \subset \Omega_k(\mathcal{K})$ if and only if $v(\rho) > 0$, the result follows immediately.
iv) This immediately follows from the identity

$$\psi(af_x - bf_y)\omega = \psi(adx + bdy).$$

■

Theorem 4.2.6 ([Be], Corollary 2, p.349)**.** *Let $\mathcal{O}$ be a plane branch. Then one has*

$$\tau(\mathcal{O}) = \ell_{\mathcal{O}}\left(\frac{\mathcal{O}}{\mathcal{C}(\mathcal{O})}\right) + \ell_{\mathcal{O}}\left(\frac{\overline{\Omega}}{\psi(\Omega)}\right).$$

Proof: Let f be an equation for $\mathcal{O}$. Since $\mathrm{j} = \langle f_x, f_y\rangle \subset \mathcal{C}(\mathcal{O})$, one has

$$\tau(\mathcal{O}) = \ell_{\mathcal{O}}\left(\frac{\mathcal{O}}{\mathrm{j}}\right) = \ell_{\mathcal{O}}\left(\frac{\mathcal{O}}{\mathcal{C}(\mathcal{O})}\right) + \ell_{\mathcal{O}}\left(\frac{\mathcal{C}(\mathcal{O})}{\mathrm{j}}\right).$$

Now, from Lemma 4.2.5 one has that

$$\ell_{\mathcal{O}}\left(\frac{\mathcal{C}(\mathcal{O})}{\mathrm{j}}\right) = \ell_{\mathcal{O}}\left(\frac{m_\omega(\mathcal{C}(\mathcal{O}))}{m_\omega(\mathrm{j})}\right) = \ell_{\mathcal{O}}\left(\frac{\overline{\Omega}}{\psi(\Omega)}\right),$$

which implies the result.

■

We will also need the following simple result from Linear Algebra (cf. [Az], Proposition 6, Chapter I):

Lemma 4.2.7. *Suppose $\mathcal{N} \subseteq \mathcal{M}$ be k-vector subspaces of $k[[t]]$ such that $\mathcal{N}$ contains all elements in $k[[t]]$ of sufficiently high order. Then $v(\mathcal{M}) \setminus v(\mathcal{N})$ is a finite set; say $v(\mathcal{M}) \setminus v(\mathcal{N}) = \{\alpha_1, \ldots, \alpha_s\}$. For each $i = 1, \ldots, s$ choose $z_i \in \mathcal{M}$ such that $v(z_i) = \alpha_i$. We have*

1. $\mathcal{M} = k\, z_1 + \cdots + k\, z_s + \mathcal{N}$;

2. $z_1, \dots, z_s$ *are linearly independent over* k *modulo* $\mathcal{N}$.

In particular, $\dim_k \Big(\mathcal{M}/\mathcal{N}\Big) = \#\Big(v(\mathcal{M}) \setminus v(\mathcal{N})\Big)$.

Notice that $\overline{\Omega} \simeq k[[t]]\,dt \simeq k[[t]]$ (as k-vector spaces). Following the notation of the preceding Lemma, set $\mathcal{M} := \overline{\Omega}$ and $\mathcal{N} := \psi(\Omega)$. Observe that $\mathcal{N}$ satisfies the required hypothesis because since $x'\,dt \in \psi(\Omega)$ we have $\mathcal{C}(\mathcal{O})\,x'\,dt \subseteq \psi(\Omega)$ where $\mathcal{C}(\mathcal{O})$ is the conductor of $\mathcal{O}$. Hence $\psi(\Omega)$ contains any element of order $\geqslant c + v(x') + 1$. So the Lemma implies that

$$\dim_k \Big(\overline{\Omega}/\psi(\Omega)\Big) = \#\Big(\mathbb{N} \setminus \Lambda(\mathcal{O})\Big) \tag{4.6}$$

As we have already seen above, in the tame case, we have $S(\mathcal{O})^* \subseteq \Lambda(\mathcal{O})$. Our next purpose is to establish, as it is known in characteristic zero (see [Za]), that in the tame case we still have

$$\#\Big(\Lambda(\mathcal{O}) \setminus S(\mathcal{O})\Big) = c - \tau(\mathcal{O}). \tag{4.7}$$

Indeed, since $S(\mathcal{O})^* \subseteq \Lambda(\mathcal{O}) \subseteq \mathbb{N}^*$ we have, using Theorem 4.2.6 and (4.6), that

$$\begin{aligned}
\#\Big(\Lambda(\mathcal{O}) \setminus S(\mathcal{O})\Big) &= \#\Big(\Lambda(\mathcal{O}) \setminus S(\mathcal{O})^*\Big) \\
&= \#\Big(\mathbb{N}^* \setminus S(\mathcal{O})^*\Big) - \#\Big(\mathbb{N}^* \setminus \Lambda(\mathcal{O})\Big) \\
&= \#\{\,\text{gaps of } S(\mathcal{O})\,\} - \dim_k \Big(\overline{\Omega}/\psi(\Omega)\Big) \\
&= \ell_{\mathcal{O}}\Big(\frac{\mathcal{O}}{\mathcal{C}(\mathcal{O})}\Big) - \tau(\mathcal{O}) + \ell_{\mathcal{O}}\Big(\frac{\mathcal{O}}{\mathcal{C}(\mathcal{O})}\Big) \\
&= c - \tau(\mathcal{O}).
\end{aligned}$$

This in particular shows, in the *tame case*, as in characteristic zero, that

$$\Lambda(\mathcal{O}) = S(\mathcal{O})^* \iff c = \tau(\mathcal{O}).$$

Example 4.2.8. *Suppose* $p = \operatorname{char} k > 0$ *and* $f = Y^n - X^m$ *with* $\gcd\{n, m\} = 1$ *and* $p \nmid nm$. *Put* $\mathcal{O} = \mathcal{O}_f$. *It is easy to check that* $f \in J(f)$. *Hence* $\tau(\mathcal{O}) = \mu(f) = c = (n-1)(m-1)$ *because* $S(\mathcal{O}) = \langle n, m\rangle$ *is tame. In particular we get* $\Lambda(\mathcal{O}) = S(\mathcal{O})^*$.

4.3 About a result of Zariski

Our next goal is to discuss a result obtained by Zariski in the case of characteristic zero (see Theorem 4 of [Za2]). We are going to show that the natural extension of Zariski's result:

If $\mathcal{O}$ is a plane branch with $S(\mathcal{O}) = \langle v_0, v_1, \ldots, v_g \rangle$ where $v_0 = n$ and $v_1 = m$ are such that $p \nmid nm$ and $\Lambda(\mathcal{O}) \setminus S(\mathcal{O}) = \emptyset$, then $\mathcal{O}$ admits an equation of the form $Y^n - X^m$ with $\gcd\{n, m\} = 1$.

fails in positive characteristic.

Zariski's strategy of the proof was to show that if $\mathcal{O}$ does not admit an equation of the form $Y^n - X^m$, then it is possible to exhibit a special differential $\omega_0 \in \Omega$ such that $v(\omega_0) \in \Lambda(\mathcal{O}) \setminus S(\mathcal{O})$. To build such a differential, choose a basis $\{x, y\}$ for the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ such that x is a transversal parameter. This choice determines, modulo a unit, an equation $f \in \mathcal{R}$, which in turn induces on $\mathcal{O}$ a discrete valuation v, so that $S(\mathcal{O}) = v(\mathcal{O}^*)$. Let c be the conductor of $S(\mathcal{O})$, then following Zariski in [Za2] or in [Za], Lemma 2.6, we may assume that after a suitable choice of the parameters x and y one gets a special *short parametrization* for $\mathcal{O}$:

$$x = t^n, \quad y = t^m + \sum_{\alpha=1}^{c/2} b_\alpha t^{\lambda_\alpha}, \quad \lambda_\alpha > m \ \text{ and } \ \lambda_\alpha, \in \mathbb{N} \setminus S(\mathcal{O}), \lambda_\alpha + n \notin n\mathbb{N} + m\mathbb{N}. \tag{4.8}$$

So, if $\mathcal{O}$ has no equation of the form $Y^n - X^m$, then $\mathcal{O}$ has a parametrization of the form (4.8) such that $b_1 \neq 0$ and $\lambda_1 + n \notin S(\mathcal{O})$ (cf. [Za2], Lemma 5)

All this remains true in positive characteristic under the assumption $p \nmid nm$.

Now, if we define the differential

$$\omega_0 = my\,dx - nx\,dy \in \Omega,$$

a direct computation shows that, in characteristic zero, $v(\omega_0) = \lambda_1 + n$ which is in $\Lambda(\mathcal{O}) \setminus S(\mathcal{O})$, as desired. The trouble is that in positive characteristic it may happen that $v(\omega_0) > \lambda_1 + n$ and $v(\omega_0) \in S(\mathcal{O})$, or worse, ω_0 may vanish, which ruins Zariski's argument in this situation.

As a matter of fact, that result of Zariski is not true in positive characteristic as shows the following example.

Example 4.3.1. *Let $f = Y^4 - X^7 - X^5Y^2 - X^{10} - X^{13}$. This is an irreducible curve of genus one. Over a field of characteristic $p = 3$ it has a primitive parametrization* *given by $x(t) = t^4$ and $y(t) = t^7 + t^{13}$. Its semigroup is equal to $\langle 4, 7\rangle$ (so it is tame) and a direct computation shows that $\tau(\mathcal{O}_f) = c(\mathcal{O}_f) = 18$. Now we claim that $\mathcal{O}_f$ is not quasi-homogeneous, that is, it is not isomorphic to $\mathcal{O}_{\tilde{f}}$ where $\tilde{f} = Y^4 - X^7$.*

Otherwise, there would exist a change of coordinates

$$\varphi(X, Y) = (aX + bY + g\,,\, cX + dY + h)$$

such that

$$fu = \tilde{f} \circ \varphi.$$

where u is a unit, $a, b, c, d \in k$ are such that $ac - bd \neq 0$ and $\mathrm{mult}(g), \mathrm{mult}(h) \geq 2$. If this is the case, evaluating the preceding parametrization on the above identity we would get

$$E := \Big(ct^4 + d(t^7 + t^{13}) + h\Big)^4 - \Big(at^4 + b(t^7 + t^{13}) + g\Big)^7 = 0 \in k[[t]].$$

By analyzing the coefficients of E of order 16 and 28 we conclude that $c = 0$ and $a^7 = d^4$, respectively. Now, write $h = h_{20}X^2 + h_{02}Y^2 + h_{11}XY + h_{30}X^3 + h_{03}Y^3 + h_{21}X^2Y + h_{12}XY^2 + hot$ (coefficients $h_{ij} \in k$) and analyzing the coefficients of order 29 of E we conclude that $h_{20} = 0$. We also get $b = 0$ looking at the terms of order 31.

Writing $g = g_{20}X^2 + g_{02}Y^2 + g_{11}XY + hot$ and analyzing the coefficients of order 32 we conclude that $d^3 h_{11} = a^6 g_{20}$. Since $\mathrm{ord}_t E > 33$ we find that $h_{30} = 0$ and, finally, $d = 0$ since $\mathrm{ord}_t E > 34$. All these constraints together imply a contradiction with the condition $ad - bc \neq 0$.

A more dramatic counterexample to Zariski's result in positive characteristic is the following branch with semigroup of genus 2.

Example 4.3.2. *Consider the branch $\mathcal{O}$ with the following parametrization*

$$x = t^4, \quad y = t^6 + t^{13}.$$

in characteristic $p = 7$. We have that $S(\mathcal{O}) = \langle 4, 6, 19\rangle$ which is tame and has conductor $c = 22$. This branch has $f = (Y^2 - X^3)^2 - 4X^8Y - X^{13}$ as an equation. A computation with Singular shows that $\mu(f) = c = \tau(f)$, that is, $\Lambda(\mathcal{O}) \setminus S(\mathcal{O}) = \emptyset$, while $\mathcal{O}$ has no quasi-homogeneous equation, since $S(\mathcal{O})$ has genus 2.

As a final remark, we observe that if we impose some restriction on the semigroup $S(\mathcal{O})$, as for instance, if the elements ℓ of $\mathbb{N} \setminus S(\mathcal{O})$ which are greater than m with

$\ell + n \notin S(\mathcal{O})$ are such that $\ell \not\equiv m \bmod p$, then the condition $\Lambda(\mathcal{O}) \setminus S(\mathcal{O}) = \emptyset$ implies that $\mathcal{O}$ has an equation of the form $Y^m - X^n$ with $\gcd(n, m) = 1$.

Just to show that there are semigroups fulfilling the above condition, take the semigroup $\langle 5, 7\rangle$ and let $p \neq 2, 3, 11$. The gaps ℓ of the semigroup greater than 7 and such that $\ell + 5 \notin \langle 5, 7\rangle$ are $8, 11, 13$ and 18. All of them are such that $\ell \not\equiv 7 \bmod p$.

APPENDIX A

Algorithm for testing null-forms

The aim of this section is to illustrate the usage of the software *Singular* ([DGPS]) to test whether a homogeneous polynomial $G \in k[Y_0, \ldots, Y_n]$ is a null-form for the Tjurina ideal $T(f) = \langle f, f_{X_1}, \ldots, f_{X_n} \rangle$, where $f \in \mathcal{R} = k[[X_1, \ldots, X_n]]$. We strongly emphasize that there is no originality here in the sense that we are only using routines already implemented by the developers and their collaborators.

With the interface of the program open, type

LIB "hnoether.lib";

This is a set of packages that you are telling *Singular* you will use. You need to declare what is the ring that you will work at. Let's say you want to work in characteristic 2, in four variables. Then type

ring r=2,(x,y,z,w),ds;

The last *ds* is the monomial ordering of power series. So, now, we introduce the power series in study. For type

*poly f=w3+x*y3+y*z3+z*x7;*

In order to check if f has or not an isolated singularity at the origin one can compute the Tjurina number $\tau(f)$ as follows:

tjurina(f);

The answer given by *Singular* is $\tau(f)$ if f has an isolated singularity at the origin and -1 otherwise. In this specific example we get $\tau(f) = 128$. Likewise one can compute the Milnor number of f:

milnor(f);

Again *Singular* computes $\mu(f)$ if it is finite and returns -1 otherwise. In our example we get $\mu(f) = \infty$. In order to compute a null-form for $T(f)$ we first introduce this ideal. For, let us first declare its generators.

poly g=diff(f,x);
poly h=diff(f,y);
poly i=diff(f,z);
poly j=diff(f,w);

Now we define $T(f)$ as being the ideal generated by the power series f, g, h, i, j. For this type

ideal T=std(f,g,h,i,j);

The command *std* tells *Singular* to compute the standard basis of $\langle f, g, h, i, j\rangle$ with respect to the given monomial ordering ds. This will make forthcoming computations easier and effective. Now we declare to *Singular* some of the ideals $\mathfrak{m}T(f)^s$. In this case we put first the ideal $\mathfrak{m}T(f)$ as follows:

*ideal mT=std(x*T+y*T+z*T+w*T);*

In order to put $\mathfrak{m}T(f)^2$ we type *ideal mT2=std(mT*T);.*

Likewise, we introduce $\mathfrak{m}T(f)^3$, $\mathfrak{m}T(f)^4$ and $\mathfrak{m}T(f)^5$ step by step as

*ideal mT3=std(mT2*T);*

*ideal mT4=std(mT3*T);*

*ideal mT5=std(mT4*T);.*

Now, to test whether a given nonzero homogeneous polynomial

$$G \in k[Y_0, Y_1, Y_2, Y_3, Y_4]$$

(of degree let's say s) is or not a null for for $T(f)$ we first evaluate G in the 5-tuple f, g, h, i, j and tell *Singular* to test if the resulting power series belongs or not to the ideal $\mathfrak{m}T(f)^s$. For example, to check whether $G = Y_3^5$ is a null form of degree 5 we evaluate G to obtain $f_z^5 = i^5 = y^5z^{10} + x^7y^4z^8 + x^{28}yz^2 + x^{35}$ and ask *Singular* to test whether this belongs to $\mathfrak{m}T(f)^5$. For this type

reduce(i5,mT5);

The result is 0 if and only if $i^5 \in \mathfrak{m}T(f)^5$ or, the same, $G \in \mathcal{N}_{T(f)}$. In this case we get a result different from zero. Hence Y_3^5 is not a null-form. Nevertheless for $G = Y_0Y_1^4$ we get $f \cdot f_x^4 = f \cdot g^4$ and this power series belong to $\mathfrak{m}T(f)^4$. Therefore $Y_0Y_1^4 \in \mathcal{N}_{T(f)}$. This implies in particular that the Milnor number $\mu(\mathcal{O}_f)$ can be computed as $\mu(f(1+\alpha x))$ for every $\alpha \neq 0$. For instance, *Singular* can make this calculation:

*milnor((1+x)*f);*

The result is 140.

Example: Here we have some examples of series $f \in k[[X, Y]]$ of which we explicitly compute a nonzero null-form of $T(f)$ with the method above and therefore, using Theorem 2.3.5, compute $\mu(\mathcal{O}_f)$.

a. Suppose $p = 3$ and let $f = Y^3 - X^{11} + X^8Y$. Then $f_X = X^7(X^3 - Y)$ and $f_Y = X^8$, so that $\mu(f) = \infty$. We check that $f_X^3 \in \mathfrak{m}T(f)^3$ so that $Y_1^3 \in \mathcal{N}_{T(f)} \setminus 0$. It follows that every unit of the form $u = 1 + aX + bY$ with $a \neq 0$ is such that $\mu(uf) = \mu(\mathcal{O}_f)$. For example, $\mu(\mathcal{O}_f) = \mu((1 + X)f) = 24$. Here we have $\mu((1 + Y)f) = 29 > \mu(\mathcal{O}_f)$, $\tau(f) = 22$ and $Z(\mathcal{N}_{T(f)}) = Z(Y_1)$.

b. Suppose $p = 3$ and let $f = Y^3 - X^{11} + X^7Y^2$. Then $f_X = X^6(X^4 + Y^2)$ and $f_Y = -X^7Y$, so that $\mu(f) = \infty$. We check that $f_Y^3 \in \mathfrak{m}T(f)^3$ so that $Y_2^3 \in \mathcal{N}_{T(f)} \setminus 0$. It follows that every unit of the form $u = 1 + aX + bY$ with $b \neq 0$ is such that $\mu(uf) = \mu(\mathcal{O}_f)$. For example, $\mu(\mathcal{O}_f) = \mu((1+Y)f) = 30$. Here we have $\mu((1+X)f) = 31 > \mu(\mathcal{O}_f)$, $\tau(f) = 24$ and $Z(\mathcal{N}_{T(f)}) = Z(Y_2)$.

c. Suppose $p = 3$ and let $f = Y^3 - X^{11} + X^8Y^2$. Then $f_X = X^7(X^3 - Y^2)$ and $f_Y = -X^8Y$, so that $\mu(f) = \infty$. We check that $f_Y^3 \in \mathfrak{m}T(f)^3$ so that $Y_2^3 \in \mathcal{N}_{T(f)} \setminus 0$. It follows that every unit of the form $u = 1 + aX + bY$ with $b \neq 0$ is such that $\mu(uf) = \mu(\mathcal{O}_f)$. For example, $\mu(\mathcal{O}_f) = \mu((1+Y)f) = 30$. Here we have $\mu((1+X)f) = 34 > \mu(\mathcal{O}_f)$, $\tau(f) = 26$ and $Z(\mathcal{N}_{T(f)}) = Z(Y_2)$.

d. Suppose $p = 3$ and let $f = Y^3 - X^{11} + X^9Y$. Then $f_X = X^{10}$ and $f_Y = X^9$, so that $\mu(f) = \infty$. We check that $f_X^3 \in \mathfrak{m}T(f)^3$ so that $Y_1^3 \in \mathcal{N}_{T(f)} \setminus 0$. It follows that every unit of the form $u = 1 + aX + bY$ with $a \neq 0$ is such that $\mu(uf) = \mu(\mathcal{O}_f)$. For example, $\mu(\mathcal{O}_f) = \mu((1 + X)f) = 27 = \tau(f)$. Here $Z(\mathcal{N}_{T(f)}) = Z(Y_1)$.

Another way for computing $\mu(\mathcal{O}_f)$ using *Singular* when you are not able to guess what would be a null-form for $T(f)$ is as follows. You declare to *Singular* that you want to work in a greater field. For example, if you want to compute $\mu(\mathcal{O}_f)$ of a plane curve it is convenient to work in $k(a, b)$ where a, b are transcendental over k and have no relation among them. It makes $1 + aX + bY$ a generic unit over k. However, to work in such an extension is computationally hard and *Singular* sometimes spend much more time performing the calculations. To use this type

LIB "hnoether.lib"; declaring packages you will use
ring r=(13,a,b),(x,y),ds; base ring with large field of constants
poly f=((y2-x3)2-x11y)2-(y2-x3)x19; polynomial you want to study
*poly u=1+ax+b*y;* your generic unit

*milnor(u*f);* computing $\mu(\mathcal{O}_f)$.

APPENDIX B

A connection with vector fields

Another interesting connection with the phenomenon of the existence of isolated hypersurface singularities $\mathcal{O}_f$ for which $\mu(f) = \infty$ is the classification of singularities of vector fields. Since the aforementioned phenomenon is a particularity of the positive characteristic setting we restrict ourselves to this case here. No attempt of originality or completeness is made: we prefer to collect here known facts and establish the connection with our approach. Moreover, all known results are about vector fields of smooth surfaces so that the connections we are able to make are restricted to singularities of plane curves.

We recall some useful and general facts about vector fields over fields of positive characteristic.

First of all, let us consider K be a field and D be a *derivation* of K, that is, a function $D : K \longrightarrow K$ satisfying, for all $f, g \in K$, that

$$D(f + g) = D(f) + D(g)$$

and

$$D(fg) = fD(g) + gD(f) \quad \text{(Leibniz rule)}.$$

Obviously the set of all derivations of K is a K-vector space which will be denoted $Der\, K$. Note that the subset of K consisting of all elements killed by D

$$K^D = \{g \in K\, ;\, D(g) = 0\}$$

is a subfield of K. We have $K^p \subseteq K^D$ from the Leibniz rule. Hence the field extension K/K^D is purely inseparable.

Now let us fix a subfield L of K such that K/L is a purely inseparable extension of degree $p = \mathrm{char} K$ and denote

$$Der_L\, K = \{D \in Der\, K\, ;\, L \subseteq K^D\}.$$

We claim first that $Der_L\, K$ is a K-vector space of dimension one. Indeed, being the extension of prime degree p there is a primitive element $x \in K \setminus L$ so that $K = L[x]$. Then each derivation D of K vanishing on L is such that $D = D(x)D_x$ where D_x is defined by $D_x(g(x)) = g'(x)$ for $g(x) \in L[x]$. This concludes the proof of the claim.

In general, a power

$$D^n = \underbrace{D \circ D \circ \cdots \circ D}_{n\,\text{times}}$$

of a derivation D of K is not a derivation anymore since we do not have the Leibniz rule for D^n: if $f, g \in K$ then

$$D^n(fg) = fD^n(g) + \sum_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} D^i(f)D^{n-i}(g) + gD^n(f).$$

However if the power is $n = p$ it also follows from the above identity that D^p is indeed a derivation of K and certainly the kernel K^{D^p} of D^p contains the kernel K^D of D.

Hence, for each $D \in Der_L\, K$, K^{D^p} also contains L and being $Der_L\, K$ a K-vector space of dimension one, we conclude that

$$D^p = hD$$

for some $h \in K$. Derivations satisfying this last property are called *p-closed derivations.*

Sending $L \longmapsto Der_L\, K$ we construct a function

$$\left\{ \begin{array}{c} \text{Subfields } K^p \subseteq L \subseteq K \\ \text{with } [K:L] = p \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} [D] \in \mathbb{P}(Der K) \\ D \text{ is } p\text{-closed} \end{array} \right\}.$$

Actually, it follows from the Galois correspondence for purely inseparable field extensions due to Jacobson [J], that this is a bijection whose inverse is given by $[D] \longmapsto K^D$.

From now on, let us consider S be a smooth surface over an algebraically closed field k and D be a *rational vector field* on S, that is, a derivation of the field $k(S)$ vanishing on k. One clearly has that $[k(S) : k(S)^p] = p^2$ and, from the above bijection, D is a p-closed derivation if and only if it does not have a trivial kernel, that is, $k(S)^p \subsetneq k(S)^D \subsetneq k(S)$.

If P is a point of S and (x, y) are local coordinates of S at P we may write

$$D = h_P \left(f_P \frac{\partial}{\partial x} + g_P \frac{\partial}{\partial y} \right)$$

where $h_P \in k(S)$ and $f_P, g_P \in \mathcal{O}_{S,P}$ are relatively prime. The functions $\{h_P\}_{P \in S}$ determine a divisor (D) in S called divisor of D. If $f_P, g_P \in \mathfrak{m}_{P,S}$ we say that D has an *isolated singularity* at P and otherwise we say that D *has only divisorial singularities* in a neighbourhood at P. We say that two p-closed vector fields D and D' on S are equivalent, writing $D \sim D'$, if there is a rational function $h \in k(S)$ such that $D = hD'$. An important characterization of vector fields having only divisorial singularities in a neighbourhood at a point is the following, due to Seshadri ([S]).

Proposition B.1.1. *Let S be a smooth surface and D a p-closed vector field on S which has only divisorial singularities in a neighbourhood of a point $P \in S$. Then in the completion $\widehat{\mathcal{O}_{S,P}}$ of the local ring of the point P there exist local parameters x, y such that $D \sim \partial/\partial y$.*

Proof: See [S] §3, Proposition 6. ∎

As an application of the preceding Proposition we have the following criterion.

Proposition B.1.2. *Let $f \in k[X, Y]$ vanishing at the origin $0 \in \mathbb{A}^2_k$ with an isolated singularity at the origin and $\mu_0(f) = \infty$, where k is an algebraically closed field of characteristic $p > 0$. Set $h = \gcd(f_X, f_Y) \in \mathfrak{m} \subset \mathcal{R} = k[[X, Y]]$. Then there exists a change of coordinates φ of $\mathcal{R}$ such that $\varphi(f) \in k[[X, Y^p]]$ if and only if f_X/h or f_Y/h does not belong to $\mathfrak{m}$.*

Proof: We consider the rational vector field of $\mathbb{A}^2_k$ given by $D_f = f_Y \frac{\partial}{\partial X} - f_X \frac{\partial}{\partial Y}$.
In order to show that D_f is a p-closed vector field it is enough, from the Galois correspondence for purely inseparable field extensions, to see that

$$k(X^p, Y^p) \subsetneq k(X, Y)^{D_f} \subsetneq k(X, Y).$$

These strictly inclusions hold because since f has an isolated singularity in the origin, then $f \notin (k[X, Y])^p = k[X^p, Y^p]$, that is, $f_X \neq 0$ or $f_Y \neq 0$. Indeed, this implies that

$f \in k(X,Y)^{D_f} \setminus k(X^p, Y^p)$ and $D_f(X) \neq 0$ or $D_f(Y) \neq 0$, that is, $X \in k(X,Y) \setminus k(X,Y)^{D_f}$ or $Y \in k(X,Y) \setminus k(X,Y)^{D_f}$.

Now, to apply Proposition B.1.1 and to prove the converse we just need to write

$$D_f = h\left(\frac{f_Y}{h}\frac{\partial}{\partial X} - \frac{f_X}{h}\frac{\partial}{\partial Y}\right)$$

where $h = \gcd(f_X, f_Y)$ in $\mathcal{R}$ to see that D_f has only divisorial singularities in a neighbourhood at the origin if and only if f_X/h or f_Y/h does not belong to $\mathfrak{m}$. On the other hand, if $f \in k[[X, Y^p]]$, then one clearly has $f_X/h = 1 \notin \mathfrak{m}$. ∎

The above Corollary does not provide the complete classification of equations having infinite Milnor number as we can see in the example below.

Example B.1.3. *When $p = 3$ the polynomial $f = X^2Y + Y^2X$ has isolated singularity at the origin since $\tau(f) = 4$. We also have $\mu(f) = \infty$, $f_X = Y(Y - X)$, $f_Y = -X(Y - X)$ and $h = \gcd(f_X, f_Y) = Y - X$. Therefore, f_X/h and f_Y/h belong to $\mathfrak{m}$ and the p-closed vector field D_f on $\mathbb{A}_k^2$ has the origin as an isolated singularity.*

The above discussion shows that the classification of singularities of p-closed vector fields on $\mathbb{A}_k^2$ can be related to the classification of algebroid plane curve singularities. However there are very few cases classified of such vector fields singularities in the literature. For more information about this subject we refer to [H] and [RS].

APPENDIX C

An example

As we mentioned before, we believe that the converse of Theorem 3.2.4 is true, in the sense that if $\mu(f) = c(f)$, then $S(f)$ is a tame semigroup, or, equivalently, if p divides any of the minimal generators of $S(f)$, then $\mu(f) > c(f)$. We include here calculations which are not very enlightening though they show, by *brute force* calculations, this converse for a *special element* of an equisingularity class of low genus (namely genus 1 and 2).

Here we include in a concise form some of the results obtained by Assi and Barile in [A-B]. See also the references therein. We use here definitions and notation introduced in the beginnings of section 3.2. In [A-B] one of the concerns of the authors was the construction of a canonical plane branch associated to a numerical semigroup $S \subseteq \mathbb{N}$ satisfying the following necessary (and sufficient) conditions to be the semigroup of values of a plane branch over k, where k is an arbitrary algebraically closed field.

Conditions: suppose that S is minimally generated by $0 < v_0 < \cdots < v_g$, so that $S = \langle v_0, \ldots, v_g \rangle$. Then there exists a plane branch f over k such that $S = S(f)$ if and only if $n_i v_i < v_{i+1}$, for all $i \in \{1, \ldots, g-1\}$.

Remark C.1.1. *We say that a sub-semigroup $S = \langle v_0, \ldots, v_g \rangle$ of $\mathbb{N}$ satisfying $n_i v_i < v_{i+1}$, for all $i \in \{1, \ldots, g-1\}$ is a strongly increasing semigroup of genus g.*

According to the Remark 3.3 [*loc.cit.*], the construction of a canonical model of branch f_S associated to the semigroup $S = \langle v_0, \ldots, v_g \rangle$ can be done inductively as follows:

1) Let $1 \leq j \leq g$. We compute the (unique) j-tuple

$$\theta^{(j)} = (\theta_0^{(j)}, \ldots, \theta_{j-1}^{(j)}) \in \mathbb{N}^j$$

satisfying $0 \leq \theta_i^{(j)} < n_i$, for $i = 1, \ldots, j-1$ and

$$\frac{v_j}{e_j} n_j = \theta_0^{(j)} \frac{v_0}{e_k} + \cdots + \theta_{k-1}^{(j)} \frac{v_{j-1}}{e_k}.$$

2) Let $G_0 := Y$ and for $1 \leq j \leq g$,

$$G_j = G_{j-1}^{n_j} - X^{\theta_0^{(j)}} Y^{\theta_1^{(j)}} \cdots G_{j-2}^{\theta_{j-1}^{(j)}}.$$

Remark C.1.2. *The construction of [A-B] provides us with an irreducible and monic (in Y) polynomial $f_S = G_g$ which we will call the Assi-Barile curve of S. Moreover we have that $S(f_S) = S$ by the construction and that f_S belongs to $\mathbb{F}_p[X, Y]$ if $\operatorname{char} k = p > 0$ and to $\mathbb{Q}[X, Y]$ if $\operatorname{char} k = 0$.*

Example C.1.3. *Let $S = [6, 8, 65]$. It is easy to check that $(e_0, e_1, e_2) = (6, 2, 1)$ and $(n_1, n_2) = (3, 2)$. Using the preceding conditions we see that S is the semigroup of values of a plane branch. According to the above algorithm:*

j=1: $\theta^{(1)} = \theta_0^{(1)}$ *and* $\frac{8}{2}3 = \theta_0^{(1)} \frac{6}{2}$. *Hence* $\theta_0^{(1)} = 4$ *and*

$$G_1 = G_0^3 - X^{\theta_0^{(1)}} = Y^3 - X^4.$$

j=2: $\theta^{(2)} = (\theta_0^{(2)}, \theta_1^{(2)})$ *with* $0 \leq \theta_1^{(2)} < n_1 = 3$ *and*

$$\frac{65}{1} 2 = \theta_0^{(2)} \frac{6}{1} + \theta_1^{(2)} \frac{8}{1}.$$

It follows that $\theta^{(2)} = (19, 2)$. *Hence,*

$$f_S = G_2 = G_1^2 - X^{\theta_0^{(2)}} Y^{\theta_1^{(2)}} = (Y^3 - X^4)^2 - X^{19} Y^2.$$

g=1:

Let $S = \langle v_0, v_1 \rangle = \langle n, m \rangle$ be the semi-group. Then it is easy to see that the Assi-Barile canonical curve is

$$f_S = f = Y^n - X^m.$$

Clearly this power series is μ-stable if and only if p does not divide n nor m and if it is the case we have $\mu(f) = c(f) = (n-1)(m-1)$.

g=2:

Now we are going to explore the situation on which $S = \langle v_0, v_1, v_2 \rangle$ is a strongly increasing semigroup of genus 2. In this case we know that there exists uniquely determined integers $a \geq 0$ and $0 \leq b < n_1$ such that

$$v_2 = a\frac{v_0}{e_1} + b\frac{v_1}{e_1}.$$

To begin with, let us say a bit more about these integers a and b.

If $b = 0$ then $a\frac{v_0}{e_1} = v_2 > n_1 v_1 > n_1 v_0$ and we get $\frac{a}{e_1} > n_1$. Hence $a > n_1 v_1 = \frac{v_0 v_1}{e_1} > \frac{v_1}{e_1} > n_1 > 1$.

If $b > 0$ we also have a finer lower bound for a: suppose that $\frac{v_1}{e_1} \geq a$. Then we would have

$$n_1 v_1 < v_2 = a\frac{v_0}{e_1} + b\frac{v_1}{e_1} \leq a\frac{v_1}{e_1} + b\frac{v_1}{e_1} = \frac{v_1}{e_1}\Big(\frac{v_0}{e_1} + b\Big).$$

Hence

$$\frac{e_0}{e_1} v_1 < \frac{v_1}{e_1}\Big(\frac{v_0}{e_1} + b\Big)$$

and since $b > 0$, then $e_0 < n_1 + b < 2n_1 = 2\frac{e_0}{e_1}$. Therefore we would have $1 < \frac{2}{e_1}$ so that $2 > e_1 = \gcd\{v_0, v_1\} > 1$ and this is clearly a contradiction. We conclude that $a > \frac{v_1}{e_1}$.

It follows that $1 < n_1 < \frac{v_1}{e_1} < a$. We see that in any case we have $a > \frac{v_1}{e_1} > 1$. This will be useful in what follows.

We are going to study the Milnor number of the Assi-Barile model of S, namely

$$f = f_S = (Y^{\frac{v_0}{e_1}} - X^{\frac{v_1}{e_1}})^{e_1} - X^a Y^b.$$

We denote $g = Y^{\frac{v_0}{e_1}} - X^{\frac{v_1}{e_1}}$. The following (already used) formula for the conductor of S (or of f) will be of use:

$$c(f) = (n_1 - 1)v_1 + (n_2 - 1)v_2 + 1 - v_0. \tag{C.1}$$

We will divide our analysis in many cases. Let $p = \text{char}k$.

Case I: p divides v_0.
If $p|v_1$ then

$$f_X = e_1 g^{e_1-1} g_X - aX^{a-1}Y^b$$

and

$$f_Y = e_1 g^{e_1-1} g_Y - bX^a Y^{b-1}.$$

But in this case $p|e_1$ so that the partial derivatives have X as a common factor (since $a > 1$ as we have already seen). Hence $\mu(f) = \infty$.

If p does not divide v_1 we have

$$f_Y = -bX^a Y^{b-1}$$

(so it is zero if $b = 0$) and, if $b > 0$,

$$f_X = e_1 g^{e_1-1} X^{\frac{v_1}{e_1}-1} - aX^{a-1}Y^b = X^{\frac{v_1}{e_1}-1}(-v_1 g^{e_1-1} - aX^{a-\frac{v_1}{e_1}}Y^b).$$

Again we get $\mu(f) = \infty$, because X is a common factor.

Case II: p does NOT divide v_0 NOR v_1.

Subcase II.1: We treat first the situation in which p does not divide a nor b which will demand more intricate calculations:

$$
\begin{aligned}
\mu(f) &= (f_X\ ,\ v_0 g^{e_1-1}Y^{\frac{v_0}{e_1}-1} - bX^aY^{b-1}) \\
&= (f_X,\ (v_0 g^{e_1-1}Y^{\frac{v_0}{e_1}-b} - bX^a)Y^{b-1}) \\
&= (f_X\ ,Y^{b-1}) + (f_X\ , v_0 g^{e_1-1}Y^{\frac{v_0}{e_1}-b} - bX^a) \\
&= (b-1)((e_1-1)\frac{v_1}{e_1} + \frac{v_1}{e_1} - 1) + (v_1 g^{e_1-1}X^{\frac{v_1}{e_1}-1} + aX^{a-1}Y^b,\ v_0 g^{e_1-1}Y^{\frac{v_0}{e_1}-b} - bX^a) \\
&= (b-1)(v_1-1) + (h,\tilde{h}) \\
&= (b-1)(v_1-1) + (X^{\frac{v_1}{e_1}-1}\ ,\ \tilde{h}) + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ \tilde{h}) \\
&= (b-1)(v_1-1) + (\frac{v_1}{e_1}-1)((e_1-1)\frac{v_0}{e_1} + \frac{v_0}{e_1} - b) + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ \tilde{h}) \\
&= (b-1)(v_1-1) + (\frac{v_1}{e_1}-1)(v_0-b) + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ \tilde{h}) \\
&= (b-1)(v_1-1) + (\frac{v_1}{e_1}-1)(v_0-b) + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ v_0(-\frac{a}{v_1}Y^bX^{a-\frac{v_1}{e_1}})Y^{\frac{v_0}{e_1}-b} - bX^a \\
&= (b-1)(v_1-1) + (\frac{v_1}{e_1}-1)(v_0-b) + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ X^{a-\frac{v_1}{e_1}}(-\frac{av_0}{v_1}Y^{\frac{v_0}{e_1}} - bX^{\frac{v_1}{e_1}})) \\
&= (b-1)(v_1-1) + (\frac{v_1}{e_1}-1)(v_0-b) + (a-\frac{v_1}{e_1})(e_1-1)(\frac{v_0}{e_1}) \\
&\qquad + (v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ \frac{av_0}{v_1}Y^{\frac{v_0}{e_1}} + bX^{\frac{v_1}{e_1}}).
\end{aligned}
$$

Hence, to conclude, we are led to calculate this last intersection number

$$\Big(v_1 g^{e_1-1} + aY^bX^{a-\frac{v_1}{e_1}}\ ,\ \frac{av_0}{v_1}Y^{\frac{v_0}{e_1}} + bX^{\frac{v_1}{e_1}}\Big).$$

To do this we parametrize the right hand side curve by

$$X = \Big(t\Big(\frac{av_0}{v_1}\Big)^{\frac{e_1}{v_0}}\Big)^{\frac{v_0}{e_1}}$$

and

$$Y = \Big(t(-b)^{\frac{e_1}{v_1}}\Big)^{\frac{v_1}{e_1}}$$

and the above intersection number can be interpreted as the order in the parameter t of the power series

$$[v_1 t^{\frac{v_0 v_1}{e_1^2}}\{(-b)^{\frac{v_0}{v_1}} - \Big(\frac{av_0}{v_1}\Big)^{\frac{v_1}{v_0}}\}]^{e_1-1} - a\Big(\frac{av_0}{v_1}\Big)^{\frac{e_1}{v_0}(a-\frac{v_1}{e_1})}(-b)^{\frac{e_1 b}{v_1}} t^{\Big(a-\frac{v_1}{e_1}\Big)\frac{v_0}{e_1}+b\frac{v_1}{e_1}}$$

and this order is
$$\frac{v_0v_1}{e_1}-\frac{v_0v_1}{e_1^2}$$
if and only if the coefficient
$$v_1\{(-b)^{\frac{v_0}{v_1}}-\Big(\frac{av_0}{v_1}\Big)^{\frac{v_1}{v_0}}\}$$
of the left hand part is non zero in k. Otherwise it would be $n_2-\frac{v_0v_1}{e_1^2}$ because
$$\frac{v_0v_1}{e_1}-\frac{v_0v_1}{e_1^2}=n_1v_1-\frac{v_0v_1}{e_1^2}<v_2-\frac{v_0v_1}{e_1^2}$$
and the coefficient of this order term is certainly non zero in view of our hypothesis on a,b. Notice that the mentioned coefficient appears when we substitute the parametrization of
$$q:=\frac{av_0}{v_1}Y^{\frac{v_0}{e_1}}+bX^{\frac{v_1}{e_1}}$$
inside g. Therefore, a necessary and sufficient condition to have
$$v_1\Big((-b)^{\frac{v_0}{v_1}}-\Big(\frac{av_0}{v_1}\Big)^{\frac{v_1}{v_0}}\Big)=0$$
is that q divides g. This is in turn equivalent to the existence of a nonzero $c\in k$ such that
$$c\frac{av_0}{v_1}=1$$
and
$$cb=-1.$$
Equivalently we would have
$$c\Big(b+\frac{av_0}{v_1}\Big)=0$$
or, the same, $0=av_0+bv_1=n_2v_2=e_1v_2\in k$ which can only occur if $p|v_2$. Thus we are led to deal with two sub-cases.

First, if $p|v_2$ we have
$$\begin{aligned}\mu(f) &= (b-1)(v_1-1)+\Big(\frac{v_1}{e_1}-1\Big)(v_0-b)+\Big(a-\frac{v_1}{e_1}\Big)(e_1-1)\Big(\frac{v_0}{e_1}\Big)\\ &+ v_2-\frac{v_0v_1}{e_1^2}\\ &= \cdots=c(f)+v_2-n_1v_1>c(f).\end{aligned}$$

However, if p does not divide v_2

$$\begin{aligned}\mu(f) &= (b-1)(v_1-1) + \Big(\frac{v_1}{e_1}-1\Big)(v_0-b) + \Big(a-\frac{v_1}{e_1}\Big)(e_1-1)\Big(\frac{v_0}{e_1}\Big) \\ &+ n_1v_1 - \frac{v_0v_1}{e_1^2} \\ &= \cdots = c(f).\end{aligned}$$

Subcase II.2: We assume here that $p|a$. In this situation the equation $v_2 = a\frac{v_0}{e_1} + b\frac{v_1}{e_1}$ tells us that $p|v_2 \Leftrightarrow p|b$. If this is so, it implies that

$$\mu(f) = (e_1g^{e_1-1}g_X,\ e_1g^{e_1-1}g_Y) = \infty.$$

On the contrary, if p does not divide b

$$\begin{aligned}\mu(f) &= (e_1g^{e_1-1}g_X\ ,\ e_1g^{e_1-1}g_Y - bX^aY^{b-1}) \\ &= (e_1-1)(g,\ e_1g^{e_1-1}g_Y - bX^aY^{b-1}) + (g_X,\ e_1g^{e_1-1}g_Y - bX^aY^{b-1}) \\ &= (e_1-1)(g,\ X^aY^{b-1}) + (g_X,\ e_1g^{e_1-1}g_Y - bX^aY^{b-1}) \\ &= (e_1-1)\Big(a(g,\ X) + (b-1)(g,Y)\Big) + (g_X,\ e_1g^{e_1-1}g_Y - bX^aY^{b-1}) \\ &= (e_1-1)\Big(a\frac{v_0}{e_1} + (b-1)\frac{v_1}{e_1}\Big) + \Big(\frac{v_1}{e_1}-1\Big)(X,\ e_1g^{e_1-1}Y^{\frac{v_0}{e_1}-1} - bX^aY^{b-1}) \\ &= (e_1-1)\Big(v_2 - n_1v_1\Big) + \Big(\frac{v_1}{e_1}-1\Big)(X,\ g^{e_1-1}Y^{\frac{v_0}{e_1}-1}) \\ &= (e_1-1)\Big(v_2 - n_1v_1\Big) + \Big(\frac{v_1}{e_1}-1\Big)\Big(\frac{v_0}{e_1} - 1 + (e_1-1)\Big(\frac{v_0}{e_1}\Big)\Big) \\ &= (e_1-1)\Big(v_2 - n_1v_1\Big) + \Big(\frac{v_1}{e_1}-1\Big)\Big(v_0-1\Big) \\ &= \cdots \\ &= c(f).\end{aligned}$$

Subcase II.3: We assume here that $p|b$. An entirely analogous argument here shows that $p|v_2 \Leftrightarrow p|a$ and if this is so, again we have $\mu(f) = \infty$. If p does not divide a the same kind of calculation shows that $\mu(f) = c(f)$.

Case III: p does NOT divide v_0 BUT $p|v_1$. In this case we have

$$f_X = -aX^{a-1}Y^b$$

and

$$f_Y = v_0 g^{e_1-1} Y^{\frac{v_0}{e_1}-1} - bX^a Y^{b-1} = Y^{b-1}\Big(v_0 g^{e_1-1} Y^{\frac{v_0}{e_1}-b} - bX^a\Big),$$

being g as before. We treat first the easiest case, namely that one in which $p|v_2$ as well.

The equation $v_2 = a\frac{v_0}{e_1} + b\frac{v_1}{e_1}$ together with the hypothesis $p|v_1$ tells us that $p|v_2 \Leftrightarrow p|a$ and this implies $f_X = 0$ so that $\mu(f) = \infty > c(f)$.

The remaining case, in which v_2 is not a multiple of p, is treated as follows: we have

$$\begin{aligned} \mu(f) &= (a-1)(X, f_Y) + b(Y, f_Y) \\ &= (a-1)\Big((e_1-1)\Big(\frac{v_0}{e_1}\Big) + \frac{v_0}{e_1} - 1\Big) + \begin{cases} \infty, & \text{if} \quad b \geq 2 \\ a, & \text{if} \quad b = 1 \\ 0, & \text{if} \quad b = 0. \end{cases} \end{aligned}$$

If $b \geq 2$ we are done.

If $b = 1$

$$\mu(f) = (a-1)(v_0-1) + a = av_0 - v_0 + 1$$

and in this case

$$\begin{aligned} \mu(f) - c(f) &= av_0 - v_0 + 1 - \Big(\frac{e_0}{e_1} - 1\Big)v_1 - \Big(\frac{e_1}{1} - 1\Big)\Big(\frac{av_0}{e_1} + \frac{v_1}{e_1}\Big) + v_0 - 1 \\ &= av_0 - \frac{e_0}{e_1}v_1 + v_1 - \Big(av_0 + v_1 - \frac{av_0}{e_1} - \frac{v_1}{e_1}\Big) \\ &= -\frac{e_0}{e_1}v_1 + a\frac{v_0}{e_1} + \frac{v_1}{e_1} \\ &= -n_1 v_1 + v_2 > 0 \end{aligned}$$

so that $\mu(f) > c(f)$.

If $b = 0$

$$\mu(f) = (a-1)(v_0-1)$$

and again we compute

$$\begin{aligned}
\mu(f) - c(f) &= av_0 - v_0 + 1 - a - \Big(\frac{e_0}{e_1} - 1\Big)v_1 - \Big(\frac{e_1}{1} - 1\Big)\Big(\frac{av_0}{e_1} + \frac{v_1}{e_1}\Big) + v_0 - 1 \\
&= e_1 v_2 - v_0 + 1 - a - n_1 v_1 + v_1 - e_1 v_2 - 1 + v_0 \\
&= -a + v_2 - n_1 v_1 + v_1 \\
&= -\frac{v_2}{n_1} + v_2 - n_1 v_1 + v_1 \\
&= v_2 - n_1 v_1 - \frac{v_2 - n_1 v_1}{n_1} \\
&= \Big(v_2 - n_1 v_1\Big)\Big(1 - \frac{1}{n_1}\Big) > 0
\end{aligned}$$

so that again we get $\mu(f) > c(f)$.

Since the cases I, II and III exhaust all the possibilities we are in conditions to state

Proposition C.1.4. *Let $S = \langle v_0, v_1, v_2\rangle$ a strongly increasing numerical semigroup of genus 2 with conductor c_S and let $f = f_S$ be its Assi-Barile canonical model. Then $\mu(f) = c(f) = c_S$ if and only if $p \nmid v_0 v_1 v_2$.*

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