Universidade Federal Fluminense

On the classification of fibrations by genus two singular curves via fibrations by elliptic curves on surfaces

Reillon Oriel Carvalho Santos

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Tese submetida ao Programa de Pós-Graduação em Matemática da Universidade Federal Fluminense como requisito parcial para a obtenção do grau de Doutor em Matemática.

Orientador: Rodrigo Salomão
Coorientador: João Hélder Olmedo Rodrigues

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Rodrigo Salomão	UFF	Orientador
João Hélder Olmedo Rodrigues	UFF	Coorientador
Abramo Hefez	UFF	Examinador
Andre Luis Contiero	UFMG	Examinador
Cecília Salgado Guimarães da Silva	UFRJ/University of Groningen-Holanda	Examinadora
Karl-Otto Stöhr	IMPA	Examinador

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Para os devidos fins, declaro **estar ciente** do conteúdo desta **versão corrigida** elaborada em atenção às sugestões dos membros da banca examinadora na sessão de defesa do trabalho, manifestando-me **favoravelmente** ao seu encaminhamento e publicação no **Repositório Institucional da UFF**.

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Rodrigo Salomão

À minha família.

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RESUMO

Em 1944 Zariski descobriu que o teorema de Bertini sobre pontos singulares variáveis não é mais verdadeiro quando passamos de um corpo de característica zero para um corpo de característica positiva. Em outras palavras, ele encontrou fibrações por curvas singulares, que só existem em característica positiva. Tais fibrações estão conectadas com muitos fenômenos interessantes. Por exemplo, a extensão da classificação de Enriques de superfícies para características positivas (Bombieri e Mumford em 1976), os contraexemplos do teorema do anulamento de Kodaira (Mukai em 2013 e Zheng em 2016) e as singularidades isoladas com número de Milnor infinito (Hefez, Rodrigues e Salomão em 2019). Neste trabalho vamos mostrar que o processo de suavização introduzido por Shimada em 1991 pode ser usado para classificar o conjunto de fibrações por curvas singulares de gênero dois - a menos de isomorfismos entre suas fibras genéricas - de modo que suas suavizações sejam fibrações elípticas em superfícies racionais. Além disso, também descreveremos os campos de vetores que podem ser usados para recuperar tais fibrações por curvas singulares via o quociente de superfícies elípticas racionais.

ABSTRACT

In 1944 Zariski discovered that Bertini's theorem on variable singular points is no longer true when we pass from a field of characteristic zero to a field of positive characteristic. In other words, he found fibrations by singular curves, which only exist in positive characteristic. Such fibrations are connected with many interesting phenomena. For instance, the extension of Enrique's classification of surfaces to positive characteristic (Bombieri and Mumford in 1976), the counterexamples of Kodaira vanishing theorem (Mukai in 2013 and Zheng in 2016) and the isolated singularities with infinity Milnor number (Hefez, Rodrigues and Salomão in 2019). In this work we are going to show that the smoothing process introduced by Shimada in 1991 can be used to classify the set of fibrations by genus two singular curves, up to isomorphism among their generic fibers, such that their smoothing are elliptic fibrations on rational surfaces. Moreover we will also describe the vector fields that can be used to recover such fibrations by singular curves via quotient of rational elliptic surfaces.

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CHAPTER 1

Introduction

In 1944 Zariski discovered in [**Z**] that Bertini's theorem on variable singular points may fail in positive characteristic. In other words, he found fibrations $f\colon X \to Y$ *by non-smooth algebraic curves* between smooth (irreducible) algebraic varieties over an algebraically closed field k of characteristic $p > 0$. According to Mumford [**Mum**], that was Zariski's main motivation to interpret generic fibers as curves over $k(Y)$, the function field of the target, and to develop two different notions of simple points on varieties defined over non-algebraically closed fields: *regular* in the sense of having a regular local ring and *smooth* meaning that the usual Jacobian criterion is satisfied.

Fibrations by non-smooth varieties are related and encode several interesting characteristic p phenomena in algebraic geometry and singularity theory. With no intention to provide an exhaustive list, we mention that: fibrations by cuspidal curves arose in the characterization of quasi-hyperelliptic case that appeared in the extension of Enriques' classification of minimal surfaces to positive characteristic (see [**BM**]); interesting connections with counterexamples of Kodaira vanishing theorem were pointed out by Mukai in [**Muk**] and by Zheng in [**Zh**], where they showed that Kodaira vanishing theorem does not work over a surface if and only if it admits a fibration by singular curves. There is also a connection with a class of isolated hypersurfaces singularities which includes those having infinity Milnor number that appeared in [**HRS**]. The existence of these fibrations by non-smooth varieties also enables us to find other geometrical constructions that never occur in characteristic zero, for instance, a covering of $\mathbb{P}^2$ with a family of singular, strange and non-classical curves, as observed in [**Sa1**] Example 1.1.

Several papers studied the classification of fibrations by singular curves from the birational perspective as [**S2**], [**S3**], [**Sa1**], [**Sa2**] and [**CS**]. All of them used the well known strategy - that in this context appeared at first in the work of Stöhr (see [**S2**]) and - which we outline below.

Let us consider $f\colon X \to Y$ a fibration by curves between smooth integral varieties over the algebraically closed field k. We avoid trivial situations assuming that f is a proper and dominant morphism, so that almost all of its fibers are complete and integral curves. A divisor dominating Y *via* f is called an *horizontal divisor* for f. It turns out that horizontal prime divisors correspond bijectively to closed points of the generic fiber X_η of f, a complete and geometrically integral algebraic curve over $k(Y)$. By means of this correspondence, the horizontal prime divisors contained in the non-smooth locus of f correspond to the non-smooth closed points of X_η, which is, in turn, a regular curve over $k(Y)$. Therefore, f is a fibration by non-smooth curves if and only if its generic fiber X_η is a regular but non-smooth curve over $k(Y)$. In this way, it is equivalent to study proper fibrations by non-smooth curves and to study their generic fibers, that is, regular but non-smooth curves that are complete and geometrically integral.

This work is devoted less to the problem of classification of generic fibers mentioned above and more to the study of an alternative approach to the classification of fibrations by singular curves using the technique of *quotient by p-closed vector fields*. This point of view was taken by Bombieri and Mumford, locally, in their extension of Enriques' classification of surfaces to positive characteristic. It also appeared as a central tool used by Takeda [**Ta**] in his construction of counterexamples of Kodaira vanishing theorem, and by Shimada [**Sh**] in the description of the phenomenon of supercuspidal families of curves on surfaces. We will give more details on this technique of quotients by vector fields in Chapter 2. Here, we give an idea of how it goes, providing an example which illustrates the relationship between vector fields and fibrations by singular curves.

Example 1.1. *Let k denote an algebraically closed field of characteristic $p > 0$, $X = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$, $Y = \mathbb{A}^1 = \operatorname{Spec} k[t]$ and $f\colon X \to Y$ be the fibration by curves induced by $f(x,y) = y^2 + x^p$. The fiber of f over each closed point $t_0 \in Y$ is the plane curve defined by the polynomial $y^2 + x^p - t_0$, which has $(t_0^{1/p}, 0)$ as its unique singular point.*

If we consider new variables Z, W and T over k related to the others by $Z = x$, $W^p = y$ and $T^p = t$, we can define $X_1 = \mathbb{A}^2 = \operatorname{Spec} k[Z,W]$, $Y_1 = \mathbb{A}^1 = \operatorname{Spec} k[T]$. Moreover, the natural morphisms $\pi : X_1 \to X$, $F : Y_1 \to Y$ and $f_1\colon X_1 \to Y_1$ defined by $\pi(Z,W) = (Z, W^p)$, $F(T) = T^p$ and $f_1(Z,W) = W^2 + Z$, respectively, make the following diagram commutative.

$$\begin{tikzcd} X \arrow[d, "f"] & X_1 \arrow[l, "\pi"'] \arrow[d, "f_1"] \\ Y & Y_1 \arrow[l, "F"'] \end{tikzcd}$$

The important part to observe in this example is that $\mathcal{O}_X(X) = k[x,y]$ coincides with the kernel of the vector field or, equivalently, the derivation $D_1 = \frac{\partial}{\partial W}$ of $\mathcal{O}_{X_1}(X_1) = k[Z,W]$. Furthermore, the pre-image by π of the horizontal locus $V(y)$ of f, containing the singularities of the general fiber of f, is $V(W)$; it is characterized by the following geometric property: for each closed point $T_0 \in Y_1$ we have $(T_0, 0) \in V(W)$ as a point of tangency *between the fiber $f_1^{-1}(T_0)$ and the curve $V(Z - T_0)$, which is an invariant curve of D_1. Notice also that in this example the fibration f_1 is generically smooth and can be considered as a* smoothing *of f.*

Shimada in [**Sh**] related fibrations by singular curves and fibrations by smooth curves on surfaces using vector fields and Frobenius morphisms of the base curves. More precisely, he proved that after applying finitely many times a process analogous to the one in above example, we can achieve this kind of smoothing for any given singular fibration on surfaces. We will use *Shimada's smoothing process* (cf. Proposition 2.2) as a tool to classify certain fibrations by singular curves.

In what follows we describe the contents of this thesis. In Chapter 2 we collect the main definitions and invariants (particularly we introduce our notion of equivalence of fibrations and recall the definition of the arithmetic genus of the general fiber) together with results of Shimada [**Sh**], Tate [**T**] and others that we shall use in our study of fibrations by singular curves. In the same Chapter 2 we place ourselves in the setting in which we could obtain our main results, namely, that of fibrations by singular curves of arithmetic genus two on surfaces over fields of characteristic three with Shimada's smoothing providing a rational elliptic surface. To this end we will rely heavily on unpublished results of Borges Neto [**BN**] that establishes a connection between the classification of such fibrations and the classification of some fibrations by elliptic curves. By lack of a widely accessible reference we decided to present his results - which constitutes the core of the function fields machinery we will use - in considerable detail in Chapter 3. Building on the results obtained in Chapter 3, in Chapter 4 we present our main results. Namely, we use a well-known result of Cossec and Dolgachev [**CD**] to reduce the study of equivalence classes of fibrations by singular curves of arithmetic genus two over fields of characteristic three with Shimada's smoothing providing a rational elliptic surface to the study of equivalence classes of fibrations obtained by blowing-up base points of certain pencils of plane cubic curves. Since the Mordell-Weil group of rational points of an elliptic surface is a birational invariant, we use it to stratify the equivalence classes of such fibrations. Under the additional assumption that the rational elliptic surface has a Mordell-Weil group of rank zero, and through a careful analysis of resolution of base points, we succeeded to provide the geometric configurations to the pairs of generators of the referred pencils of plane cubics (cf. Theorem 4.6). It turns out that the restriction obtained in the case of Mordell-Weil rank zero is surprisingly strong: they allowed us to deduce that the corresponding stratum has dimension zero. Indeed we prove that every fibration by singular curves of arithmetic genus two over fields of characteristic three with Shimada's smoothing being a rational elliptic surface of rank zero possess as smoothing a fibration obtained by resolving base points of a unique and explicit pencil of plane cubics, up to our equivalence relation (cf. Theorem 4.7). For the sake of completeness we include an example (see Example 4.9) showing that the same phenomenon does not propagate to the case where the Mordell-Weil rank of the rational elliptic surface is one. As a result of the description of Theorem 4.7, mixing with results on function fields, we obtain an explicit 3-closed vector field on the projective plane - with the prescribed tangency divisor with the general member of such pencils - whose quotients produce our fibrations by singular curves of arithmetic genus two.

CHAPTER **2**

Smoothing and classification of fibration by singular curves

In this chapter we recall the fundamentals of Shimada's work [**Sh**] which are going to play an essential role in our approach, as explained in the Introduction.

Unless mentioned otherwise, we always work with varieties, surfaces and curves being integral k-schemes of finite type over an algebraically closed field k of characteristic $p > 0$. In particular, morphisms are k-morphisms. By technical reasons we will mean by a *fibration by (non-smooth or singular) curves* $f\colon X \to Y$ a dominant morphism between integral and complete varieties such that the general fiber is a smooth (non-smooth or singular) integral curve though the total space X is smooth after eventually restricting the base to a dense open subset.

Definition 2.1. *Two fibrations $f\colon X \to Y$ and $f'\colon X' \to Y'$ are said to be equivalent when there are birational maps $\sigma\colon Y \to Y'$ and $\varphi\colon X \to X'$ such that the following natural diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\sigma"] & Y' \end{tikzcd}$$

If this is the case we denote $f \sim f'$.

If η and η' are the generic points of Y and Y' respectively, then the generic fibers X_η and $X'_{\eta'}$ of f and f', respectively, are complete and regular algebraic curves over the fields $k(Y)$ and $k(Y')$ respectively. We notice that $f\colon X \to Y$ and $f'\colon X' \to Y'$ are equivalent if and only if their generic fibers X_η and $X'_{\eta'}$ are isomorphic curves, that is, the field extensions $k(X)|k(Y)$ and $k(X')|k(Y')$ are isomorphic over k.

Let us consider $f_1\colon S_1 \to C_1$ be a fibration by curves from a surface S_1 onto a curve C_1. Let D_1 be a rational *vector field* on S_1, that is, a k-derivation of the field $k(S_1)$ of rational functions on S_1. We say that another vector field D on S_1 is *equivalent* to D_1 when $D = hD_1$ for some $h \in k(S_1) \setminus \{0\}$. It is possible to see that the composition $D_1^p = D_1 \circ \cdots \circ D_1$ of D_1 with itself p-times is also a vector field on S_1. We assume that D_1 is *p-closed*, that is, $D_1^p = hD_1$ for some $h \in k(S_1)$.

The variety $S_1^{D_1}$, which is equal to S_1 as a topological space, with the new structure sheaf

$$\mathcal{O}_{S_1^{D_1}}(U) = \{g \in \mathcal{O}_{S_1}(U) \mid D_1(g) = 0\},$$

where U runs over the open subsets of S_1, is called the *quotient* of S_1 by the vector field D_1. It is not difficult to prove that $S_1^{D_1}$ is normal if S_1 is so. The inclusions $\mathcal{O}_{S_1^{D_1}}(U) \subset \mathcal{O}_{S_1}(U)$ induce a map

$$\pi^{D_1} \colon S_1 \to S_1^{D_1}$$

which is purely inseparable of degree p, since it is an homeomorphism of surfaces and $k(S_1)$ is a purely inseparable field extension of $k(S_1^{D_1}) = k(S_1)^{D_1} := \{g \in k(S_1) \mid D_1(g) = 0\}$ of degree p.

If we consider $\{x_P, y_P\}$ a local coordinate system at each smooth point P at S_1, we can write

$$D_1 = h_P \left(f_P \frac{\partial}{\partial x_P} + g_P \frac{\partial}{\partial y_P} \right)$$

where $h_P \in k(S_1)$ and f_P, g_P are relatively prime elements in $\mathcal{O}_{S_1,P}$. The *divisor* (D_1) associated to D_1 is defined by the functions $\{h_P\}_{P \in S_1}$. We say that P is an *isolated singularity* of D_1 when $f_P(P) = g_P(P) = 0$ and, otherwise, we say that D_1 *has only divisorial singularities* in a neighborhood of P. When D_1 has only divisorial singularities at any point of S_1 we say that D_1 *has only divisorial singularities*. We say that a curve B on S_1 is *invariant* by D_1 if $D_1(f) \in f \cdot \mathcal{O}_{S_1,P}$ for any point $P \in B$, where f is a local equation of B at P.

Proposition 2.2 (Shimada)**.** *Let us consider k an algebraically closed field of characteristic $p > 0$ and $f \colon S \to C$ be a fibration by singular curves, where S is a normal surface and C is a smooth curve over k. If S_1 is the normalization of $S \times_C C^{(1/p)}$, then there exists a p-closed vector field D_1 - uniquely determined up to equivalence - on S_1 such that the diagram*

$$\begin{tikzcd} S = S_1^{D_1} \arrow[d, "f"] & S_1 \arrow[l, "\pi^{D_1}"'] \arrow[d, "f_1"] \\ C & C_1 \arrow[l, "F_{C^{(1/p)},k}"'] \end{tikzcd}$$

commutes, where $C_1 = C^{(1/p)}$ and $F_{C^{(1/p)},k}$ is the relative Frobenius map of $C^{(1/p)}$. Moreover, after applying this process finitely many times we obtain a commutative diagram

$$\begin{tikzcd} S = S_1^{D_1} \arrow[d, "f"] & S_1 = S_2^{D_2} \arrow[l, "\pi^{D_1}"'] \arrow[d, "f_1"] & \cdots \arrow[l, "\pi^{D_2}"'] & S_n \arrow[l, "\pi^{D_n}"'] \arrow[d, "f_n"] \\ C & C_1 \arrow[l, "F_{C^{(1/p)},k}"'] & \cdots \arrow[l] & C_n \arrow[l, "F_{C_{n-1}^{(1/p)},k}"'] \end{tikzcd} \tag{2.1}$$

such that $f_n \colon S_n \to C_n$ is a fibration by (generically smooth) curves.

Definition 2.3. *Given a fibration $f \colon S \to C$ in the setting of the Proposition 2.2, we say that $f_n \colon S_n \to C_n$ is the Shimada's smoothing of f when n is the first integer such that f_n is a fibration by (generically smooth) curves.*

The previous result suggests the following problem: "Is it possible to use Shimada's process of smoothing to classify fibrations by singular curves?". With this problem in mind we observe that since Shimada's process is intrinsic, equivalent fibrations must have equivalent smoothings. Hence a possible stratification would be by discrete data which are invariant under our equivalence relation, the most prominent being the arithmetic genus p_a of the generic fiber.

Since the normalization of $(S_i)_{\eta_i} \otimes_{k(C_i)} k(C_i)^{1/p}$ is equal to $(S_{i+1})_{\eta_{i+1}}$ where η_i is the generic point of C_i, we can apply Corollary 3.2 in [**S1**] and a result due to Rosenlicht in [**R**] pg. 182 to obtain that $p_a((S_{i+1})_{\eta_{i+1}}) > p_a((S_i)_{\eta_i})$ if and only if f_i is a fibration by singular curves.

For simplicity we will study the case where $f_1 \colon S_1 \to C_1$ is already the smoothing of the fibration by singular curves $f \colon S \to C$. This can be thought as if we are going from h_{n-1} to h_n, where h_n is the Shimada's smoothing of a fibration by singular curves h. The simplest case with this setup is when the arithmetic genera g and g_1 of the generic fibers of f and f_1, respectively, are equal to 1 and 0. However, the equivalence class of f_1 does not determine the equivalence class of f, as we can see in [**Q**] Proposition 2(4). Indeed, Queen has shown in [**Q**] that there are non-isomorphic regular and non-smooth curves with $g = 1$ and $g_1 = 0$, whose smoothings have a rational point, that is they are isomorphic to the projective line. On the other hand, [**BN**] Corollary 2 pg. 34 or Chapter 3 Proposition 3.12, shows that the simplest case where we obtain a correspondence between the equivalence classes is when $g = 2$, $g_1 = 1$ and the characteristic of k is 3. Fibrations with this feature will be called *absolutely elliptic fibrations* by singular curves of arithmetic genus 2.

In this work we will start to describe the set of equivalence classes of absolutely elliptic fibrations by singular curves of genus 2, over an algebraically closed fields of characteristic 3. Actually we are going to study fibrations by singular curves $f \colon S \to C$ such that the total spaces S_1 of the smoothings f_1 in Shimada's diagram are rational surfaces. Therefore, C and C_1 are the projective lines and, locally, they are the affine lines $\mathbb{A}^1_t = \operatorname{Spec} k[t]$ and $\mathbb{A}^1_T = \operatorname{Spec} k[T]$, respectively, with $T^3 = t$. Summing up, we are interested in describing the set $\mathscr{H}$ of equivalence classes of absolutely elliptic fibrations by singular curves of genus 2 with the prescribed restrictions.

CHAPTER **3**

Absolutely elliptic regular curves of genus two

In this chapter we will study the generic fiber of absolutely elliptic fibrations by singular curves of genus two. As we will be concerned with birational invariants of these curves, we will actually study them with a function fields point of view. Actually, most results presented here are not ours and were obtained originally by Borges Neto in his unpublished PhD Thesis [**BN**]. For the sake of completeness we will present proofs.

Let us consider K a non-algebraically closed field of positive characteristic p and C be a regular complete and geometrically integral algebraic curve over K. Hence, K is algebraically closed in $K(C)$ and $K(C)|K$ is separably generated. By definition C is said to be non-smooth when $\overline{C} := C \otimes_K \overline{K}$ is a non-regular curve over $\overline{K}$. Since the arithmetic genus is invariant under base field extensions, it follows that C is non-smooth if and only if $p_g(C) = p_a(C) = p_a(\overline{C}) > p_g(\overline{C})$, where p_a and p_g stand by the arithmetic and geometric genera of the indicated curves, respectively. We recall that

$$p_a(\overline{C}) - p_g(\overline{C}) = \sum \delta_P \tag{3.1}$$

where $\delta_P = \dim_{\overline{K}} \frac{\widetilde{\mathcal{O}_{\overline{C},P}}}{\mathcal{O}_{\overline{C},P}}$, P runs over the points of $\overline{C}$ and $\widetilde{\mathcal{O}_{\overline{C},P}}$ is the integral closure of $\mathcal{O}_{\overline{C},P}$ in $\overline{K}(\overline{C})$.

In this chapter we will study regular curves C with $p_a(C) = 2$ and $p_g(\overline{C}) = 1$. Such curves are called *absolutely elliptic* regular curves of genus two. It follows by Tate's upper bound (see [**T**]) that $p \leq 2p_a(C) + 1 = 5$. Therefore such phenomenon can occur only when we are working over fields of characteristic two, three or five. More precisely, it is shown in [**BN**] and [**CS**] that absolutely elliptic curves can occur only in characteristic two or three. In this work we will focus only in the case of characteristic three.

Since genus two curves are hyperelliptic curves, it follows from Lemma 3.6.1 and Corollary 3.6.3 in [**G**] that a genus two curve C over K is birationally equivalent to an affine plane curve over K given by the equation

$$Y^2 = f(X)$$

where $f(X)$ is a square free polynomial in $K[X]$ of degree 5 or 6. Both degrees can be obtained from one another by means of isomorphisms of $K(C)|K$. From now on we will describe polynomials $f(X)$ with degree 6 for which the corresponding curves are absolutely elliptic.

We notice that $f(X)$ is a separable polynomial in $\overline{K}[X]$ if and only if the affine plane curve over $\overline{K}$ given by $Y^2 = f(X)$ is regular. This is, in turn, equivalent to the smoothness of the affine plane curve over K given by $Y^2 = f(X)$. Therefore, regular but non-smooth curves C of genus two over K are birationally equivalent to an affine plane curve over K given by the equation $Y^2 = f(X)$ where $f(X)$ is a square free polynomial in $K[X]$ of degree 6 and is inseparable as a polynomial in $\overline{K}[X]$.

Proposition 3.1. *Let us consider K be a field of characteristic three and C be a genus two regular and geometrically integral curve over K. Then, C is absolutely elliptic if and only if it is birationally equivalent to an affine plane curve given by $Y^2 - f(X)$, where $f(X) = (a_0X^3 - a_1)(b_0X^3 + b_1X^2 + b_2X + b_3)$ with $a_0, a_1, b_0, b_1, b_2, b_3 \in K$, $a_0b_0 \neq 0$, $\frac{a_1}{a_0} \notin K^3$ and $b_0X^3 + b_1X^2 + b_2X + b_3$ separable.*

Proof. If $f'(X) = 0$, then $f(X) = a_0X^6 + a_1X^3 + a_2$ with $a_0, a_1, a_2 \in K$ and $a_0 \neq 0$. Hence, in $\overline{K}[X]$, $f(X)$ can be factorized as $a(X - b)^3(X - c)^3$. We notice that $b \neq c$ because, otherwise, $f(X)$ would not be a square free polynomial in $K[X]$. In this case $\overline{C}$ has two singular points and, from (3.1), $p_g(\overline{C}) = 0$. Therefore, we can restrict ourselves to the case $f'(X) \neq 0$.

We notice firstly that $f(X)$ can not be irreducible, because otherwise, $f(X)$ would be separable implying the smoothness of C. Hence $f(X) = g(X)h(X)$ with $g(X)$ and $h(X)$ having $\gcd(g(X), h(X)) = 1$ in $K[X]$. We consider $q(X)$ being an irreducible common factor of $f(X)$ and $f'(X)$. Since $f(X)$ is square free, we may assume without loss of generality that $g(X) = q(X)$. From $f'(X) = q'(X)h(X) + q(X)h'(X)$ we obtain that $q(X)$ also divides $q'(X)$. Hence $q'(X) = 0$ or equivalently $q(X) \in K[X^3]$. Therefore, $f(X) = (a_0X^3 + a_1)(b_0X^3 + b_1X^2 + b_2X + b_3)$ with $a_0, a_1, b_0, b_1, b_2, b_3 \in K$, $a_0b_0 \neq 0$. We also have $\frac{a_1}{a_0} \notin K^3$ since $a_0X^3 + a_1 = q(X)$ is irreducible. To finish the proof, we notice that $f'(X) \neq 0$ implies that $(b_0X^3 + b_1X^2 + b_2X + b_3)' \neq 0$, that is, $b_0X^3 + b_1X^2 + b_2X + b_3$ has no triple root in $\overline{K}$. It is not difficult to prove that if $b_0X^3 + b_1X^2 + b_2X + b_3$ has a double root a and a simple root b, both in $\overline{K}$, then they must belong to K. However, in this case $f(X)$ would not be a square free polynomial. Therefore, $b_0X^3 + b_1X^2 + b_2X + b_3$ must be a separable polynomial. □

It is possible to use the previous result to give a normal form for absolutely elliptic genus two regular curves C over a field K of characteristic three, as we will do below. Notice that C is birationally equivalent to an affine plane curve given by

$$Y^2 - (X^3 - \alpha)(f_0X^3 + f_1X^2 + f_2X + f_3) \tag{3.2}$$

with $\alpha, f_0, f_1, f_2, f_3 \in K$, $\alpha \notin K^3$ and $f_0X^3 + f_1X^2 + f_2X + f_3$ separable. In this way $K(C) = K(x, y)$ where x and y are the residue classes of X and Y modulo $Y^2 - (X^3 - \alpha)(f_0X^3 + f_1X^2 + f_2X + f_3)$, respectively.

Let us consider $L = K(\beta)$ where $\beta \in \overline{K}$ is such that $\beta^3 = \alpha$. We can write

$$y^2 = (x - \beta)^3(f_0(x - \beta)^3 + f_1(x - \beta)^2 + (f_2 - \beta f_1)(x - \beta) + (f_3 + f_2\beta + f_1\beta^2 + f_0\beta^3))$$

in $L(C \otimes_K L) = L \cdot K(C) = L(x, y)$. By considering $z = 1/(x - \beta)$ and $w = z^3y$ we obtain that $L(C \otimes_K L) = L(z, w)$ with z and w satisfying the identity

$$w^2 = a_0z^3 + a_2z^2 + a_4z + a_6 \tag{3.3}$$

where $a_0 = f_3 + f_2\beta + f_1\beta^2 + f_0\beta^3$, $a_2 = f_2 - \beta f_1$, $a_4 = f_1$ and $a_6 = f_0$. Therefore, $C \otimes_K L$ is an elliptic curve over L with Weierstrass form given by the above relation between z and w. We refer the reader to [**Si**] and [**La**] for results about elliptic curves over non-algebraically closed fields. The discriminant associated to the Weierstrass form (3.3) is

$$\Delta = -a_0a_4^3 + a_2^2a_4^2 - a_2^3a_6 = -f_3f_1^3 + f_2^2f_1^2 - f_2^3f_0$$

which is the discriminant of the separable polynomial $f_0X^3 + f_1X^2 + f_2X + f_3$. Hence it is nonzero and the elliptic curve is smooth.

In the following we derive useful properties about some coefficients of the Weierstrass form (3.3) and its j-invariant.

Lemma 3.2. *If $w^2 = a_0z^3 + a_2z^2 + a_4z + a_6$ is a Weierstrass form obtained as in (3.3), then $a_2 \neq 0$ and $a_0 \in L \setminus K$.*

Proof. By construction we have that L is a K-vector space with base $\{1, \beta, \beta^2\}$. Since $f_3 + f_0\beta^3 \in K$ it follows that $f_3 + f_2\beta + f_1\beta^2 + f_0\beta^3 \in K$ if and only if there exists $g \in K$ such that $g + f_2\beta + f_1\beta^2 = 0$, which is possible only when $g = f_2 = f_1 = 0$. By the same reason we have $a_2 = f_2 - f_1\beta = 0$ only when $f_2 = f_1 = 0$. But this is not possible because, otherwise, $f_0X^3 + f_1X^2 + f_2X + f_3 = f_0X^3 + f_3$ would not be separable. □

The j-invariant associated to the Weierstrass form (3.3) is given by

$$j = \frac{a_2^6}{a_0^2\Delta} = \frac{(f_2^3 - f_1^3\beta^3)^2}{(f_3 + f_2\beta + f_1\beta^2 + f_0\beta^3)^2(-f_3f_1^3 + f_2^2f_1^2 - f_2^3f_0)} \in L.$$

Hence we have the following consequence of the last lemma.

Corollary 3.3. *If $w^2 = a_0z^3 + a_2z^2 + a_4z + a_6$ is a Weierstrass form obtained as in (3.3), then $j \in L \setminus K$ and $L = K(j)$.*

Proof. Since $j = a_2^6/a_0^2\Delta$ and $a_2^6, \Delta \in K$, then $j \in K$ would imply that $a_0^2 \in K$. In this case, $[K(a_0) : K]$ would be a common divisor between 2 and $[L : K] = 3$, that is, $[K(a_0) : K] = 1$. But this is not possible by the previous lemma. Since $j \in L \setminus K$ it follows immediately that $L = K(j)$. □

Remark 3.4. *Since $[L : K] = 3$, it follows from the previous corollary that $K(\beta) = L = K(j)$. We can write j with respect to the base $\{1, \beta, \beta^2\}$ as follows.*

$$j = \frac{a_2^6}{a_0^2\Delta} = \frac{a_2^6}{a_0^3\Delta}a_0 = g(f_3 + f_0\beta^3) + gf_2\beta + gf_1\beta^2$$

where $g = \frac{a_2^6}{a_0^3\Delta} \in K \setminus \{0\}$.

We notice that we can write $j = \frac{a\beta+b}{c\beta+d}$ where $ad - bc \neq 0$. Indeed, this is possible if and only if $(g(f_3 + f_0\beta^3) + gf_2\beta + gf_1\beta^2)(c\beta + d) = a\beta + b$, or equivalently, if and only if a, b, c and d form a solution for the system

$$\begin{cases} -a + c_0c + c_1d &= 0 \\ -b + c_2\beta^3c + c_0d &= 0 \\ c_1c + c_2d &= 0 \end{cases}$$

where $c_0 = g(f_3 + f_0\beta^3)$, $c_1 = gf_2$ and $c_2 = gf_1$. Such system admits $a = c_0c_2 - c_1^2$, $b = c_2^2\alpha - c_0c_1$, $c = c_2$ and $d = -c_1$ as solution. Moreover, $ad - bc = c_1^3 - c_2^3\beta^3 = g^3a_2^3 \neq 0$ from the last lemma.

Now we present a normal form for absolutely elliptic curves of genus two.

Proposition 3.5 (Borges Neto)**.** *Let C be a regular, complete and geometrically integral curve of genus two over a field K, of characteristic 3, birational to an affine plane curve given by (3.2). Then it is birational to the affine plane curve given by the polynomial*

$$Y^2 - h(X^3 - j^3)(X^3 - j^3X - j^3)$$

in $K[X, Y]$, with $h = (\alpha f_1^3 - f_2^3)\Delta \in K$, $j \in K^{1/3} \setminus K$ where Δ and j being the discriminant and j-invariant of the Weierstrass form (3.3), respectively. In this case, if we denote $L = K(j)$, then $C \otimes_K L$ is birational to the affine smooth elliptic curve defined by the polynomial

$$W^2 + h(j^4Z^3 + j^3Z^2 - 1)$$

in $L[Z, W]$, which admits a rational point at infinity.

Proof. As we have already seen $K(C) = K(x, y)$ where $y^2 = (x^3 - \alpha)(f_0x^3 + f_1x^2 + f_2x + f_3)$. If we extend the base field K to L, then we will get an elliptic curve with Weierstrass form $w^2 = a_0z^3 + a_2z^2 + a_4z + a_6$ where $a_0 = f_3 + f_2\beta + f_1\beta^2 + f_0\beta^3$, $a_2 = f_2 - \beta f_1$, $a_4 = f_1$ and $a_6 = f_0$. It follows from Corollary 3.3 that $j \neq 0$.

By Theorem 3.6.4 in [**G**], $K(x)$ is the unique subfield of $K(x, y)$ of index 2. In this way we will produce an automorphism of $K(x, y)$ induced by an automorphism of $K(x)$. Let us consider the automorphism φ of $K(x)$ defined by $\varphi(x) = \frac{Ax+B}{Cx+D}$ where $A = d$, $B = -b$, $C = -c$, $D = a$ and a, b, c and d are defined as in the previous remark. By applying φ on $(x^3 - \alpha)(f_0x^3 + f_1x^2 + f_2x + f_3)$ we obtain

$$\frac{(A^3 - \alpha C^3)\Delta g^3}{(Cx + D)^6}(x^3 - j^3)(x^3 - j^3x - j^3) = \frac{(\alpha f_1^3 - f_2^3)\Delta g^6}{(Cx + D)^6}(x^3 - j^3)(x^3 - j^3x - j^3)$$

where $g = \left(\frac{-\Delta}{a_2^5}\right)^3$. Therefore, we have $K(x, y) = K(x, y')$ where $y' = (Cx + D)^3y/g^3$ satisfies

$$y'^2 = h(x^3 - j^3)(x^3 - j^3x - j^3)$$

with $h = (\alpha f_1^3 - f_2^3)\Delta$. □

Now we are going to study the relation between birational classes of absolutely elliptic regular curves of genus two and those of corresponding elliptic curves. In contrast with the general case, it is shown by Borges Neto in [**BN**] that the birational class of a regular but non-smooth curve is determined by that of its smoothing.

Remark 3.6. *Let us consider E and E_1 two elliptic curves over a field F of characteristic three given, respectively, by the Weierstrass forms*

$$w^2 = a_0z^3 + a_2z^2 + a_4z + a_6 \text{ and } w_1^2 = b_0z_1^3 + b_2z_1^2 + b_4z_1 + b_6, \tag{3.4}$$

*with $a_2b_2 \neq 0$. If we apply the isomorphism of E given by $z = \frac{z_2}{a_0} + \frac{a_4}{a_2}$, $w = \frac{w_2}{a_0}$ then we can normalize $a_4 = 0$ and $a_0 = 1$. After doing the same kind of normalization for E_1 we can use the isomorphism described in paragraph 2, pg. 301, of [**La**] to obtain that E and E_1 are isomorphic if and only if $j = j_1$ and $\frac{a_2}{b_2} = u^2 \in F^2$. Moreover, the isomorphism between E and E_1 is induced by $\varphi\colon F(z, w) \to F(z_1, w_1)$ given by*

$$\varphi(z) = u^2z_1 \text{ and } \varphi(w) = u^3w_1.$$

In the particular case where the above Weierstrass forms are as in Proposition 3.5 we can simplify the isomorphisms between the elliptic curves as follows.

Remark 3.7. *Let us consider E and E_1 two elliptic curves over a field F of characteristic three given, respectively, by the Weierstrass forms*

$$w^2 = -h(j^4z^3 + j^3z^2 - 1) \textit{ and } w_1^2 = -h_1(j_1^4z_1^3 + j_1^3z_1^2 - 1), \tag{3.5}$$

with $h, h_1, j, j_1 \in F \setminus \{0\}$. By the above remark we obtain that E and E_1 are isomorphic if and only if $j = j_1$ and $\frac{h}{h_1} = u^2 \in F^2$. Moreover, the isomorphism between E and E_1 is induced by $\varphi\colon F(z, w) \to F(z_1, w_1)$ given by

$$\varphi(z) = z_1 \textit{ and } \varphi(w) = uw_1.$$

The following proposition characterizes when two absolutely elliptic genus two curves are isomorphic in terms of their normal forms given in Proposition 3.5.

Proposition 3.8 (Borges Neto)**.** *Let us consider C and C_1 two absolutely elliptic curves of genus two over a field K, of characteristic three, with normal forms $Y^2 - h(X^3 - j^3)(X^3 - j^3X - j^3)$ and $Y_1^2 - h_1(X_1^3 - j_1^3)(X_1^3 - j_1^3X_1 - j_1^3)$, respectively, with $j, j_1 \in K^{1/3} \setminus K$. Then C and C_1 are isomorphic if and only if $j = j_1$ and $\frac{h}{h_1} \in K^2$. In particular, C and C_1 are isomorphic if and only if the elliptic curves obtained from their respective base extensions, as in Proposition 3.5, are isomorphic.*

Proof. If C and C_1 are isomorphic then the elliptic curves $C \otimes_K K^{1/3}$ and $C_1 \otimes_K K^{1/3}$ are also isomorphic. From Remark 3.7 we obtain $j = j_1$ and $\frac{h}{h_1} \in \left(K^{1/3}\right)^2 \cap K = K^2$.

Conversely, if $j = j_1$ and $\frac{h}{h_1} = u^2 \in K^2$, then $\varphi\colon K(x, y) \to K(x_1, y_1)$ given by

$$\varphi(x) = x_1 \text{ and } \varphi(y) = uy_1$$

is the required isomorphism of function fields. □

In order to identify generic fibers of fibrations we will need a slightly more flexible notion of isomorphisms of curves naturally adapted from the notion of equivalent fibrations, where we need to allow automorphisms of targets (see Definition 2.1). The facts presented from now on reflect such flexibility that was not needed in [**BN**].

Definition 3.9. *We say that two regular, complete and geometrically integral curves $C|K$ and $C_1|K_1$, over the fields K and K_1 respectively, are isomorphic when their function fields $K(C)|K$ and $K_1(C_1)|K_1$ are isomorphic. In other words, when there are isomorphisms of fields $\sigma\colon K \to K_1$ and $\widetilde{\sigma}\colon K(C) \to K_1(C_1)$ commuting the following diagram.*

$$\begin{tikzcd} K(C) \arrow[r, "\widetilde{\sigma}"] \arrow[d, dash] & K_1(C_1) \arrow[d, dash] \\ K \arrow[r, "\sigma"] & K_1 \end{tikzcd}$$

We notice that the above proposition and the last two remarks remain working with the following modification.

Remark 3.10. *Two elliptic curves $E|F$ and $E_1|F_1$ over fields of characteristic three given, respectively, by the Weierstrass forms as in 3.4 (with $a_i \in F$, $b_j \in F_1$ and $a_2b_2 \neq 0$) are isomorphic if and only if there exists an isomorphism $\sigma\colon F \to F_1$ such that $\sigma(j) = j_1$ and $\frac{\sigma(a_2)}{b_2} = u^2 \in F_1^2$. Moreover, the isomorphism $\widetilde{\sigma}\colon F(E) \to F_1(E_1)$ is given by $\widetilde{\sigma}(z) = u^2 z_1$ and $\widetilde{\sigma}(w) = u^3 w_1$.*

Remark 3.11. *Two elliptic curves $E|F$ and $E_1|F_1$ over fields of characteristic three given, respectively, by the Weierstrass forms as in 3.5 (with $h, j \in F \setminus \{0\}$, $h_1, j_1 \in F_1 \setminus \{0\}$ are isomorphic if and only if there exists an isomorphism $\sigma\colon F \to F_1$ such that $\sigma(j) = j_1$ and $\frac{\sigma(h)}{h_1} = u^2 \in F_1^2$. Moreover, the isomorphism $\widetilde{\sigma}\colon F(E) \to F_1(E_1)$ is given by $\widetilde{\sigma}(z) = z_1$ and $\widetilde{\sigma}(w) = uw_1$.*

Proposition 3.12. *Two absolutely elliptic curves $C|K$ and $C_1|K_1$ of genus two over the fields of characteristic three K and K_1, respectively, are isomorphic if and only if the elliptic curves obtained from their respective base extensions, as in Proposition 3.5, are isomorphic.*

Next results characterize elliptic curves that are smoothings of regular but non-smooth curves of genus two.

Lemma 3.13. *Let F be a field of characteristic three and let $E|F$ be a smooth elliptic curve with Weierstrass form $w^2 = a_0z^3 + a_2z^2 + a_4z + a_6$ where $j = \frac{a_2^6}{a_0^2\Delta} \neq 0$. Then E is isomorphic to the elliptic curve $E_1|F$ given by the Weierstrass form*

$$w_1^2 = -g(j^4z_1^3 + j^3z_1^2 - 1)$$

where $g = -\frac{a_2^3}{j^3a_0^6} = \left(\frac{-\Delta}{a_2^5}\right)^3$.

Proof. Indeed we just need to consider the isomorphism $E_1 \xrightarrow{\sim} E$ induced by $\varphi\colon F(z,w) \to F(z_1,w_1)$ such that $\varphi(z) = a + bz_1$ and $\varphi(w) = ew_1$, where $a = \frac{a_4}{a_2}$, $b = j\frac{a_2}{a_0}$ and $e = a_0^2 j$. □

Proposition 3.14. *Let F be a field of characteristic three and let $E|F$ be a smooth elliptic curve with Weierstrass form $w^2 = a_0z^3 + a_2z^2 + a_4z + a_6$ where $a_0 \neq 0$. Then there exists an absolutely elliptic curve $C|F^3$ of genus two such that $C \otimes_{F^3} F$ is isomorphic to E if and only if the j-invariant of E belongs to $F \setminus F^3$.*

Proof. If $j \in F \setminus F^3$, then by Lemma 3.13 we have that E is isomorphic to an elliptic curve with Weierstrass form given by $w_1^2 = -g(j^4z_1^3 + j^3z_1^2 - 1)$ where $g = \left(-\frac{\Delta}{a_2^5}\right)^3 \in F^3$, that is, E is isomorphic to the extension of the plane affine curve given by the polynomial $Y^2 - g(X^3 - j^3)(X^3 - j^3X - j^3)$ in $F^3[X,Y]$. This curve is regular but non-smooth of genus 2 since $j \in F \setminus F^3$.

Now if C is a regular but non-smooth curve over F^3 then, by the above proposition, C is birational to an affine curve given by the polynomial $Y^2 - h(X^3 - j^3)(X^3 - j^3X - j^3) \in F^3[X,Y]$ where $j \notin F^3$ is the j-invariant of the smooth elliptic curve $C \otimes_{F^3} F^3(j)$. Hence $C \otimes_{F^3} F$ is an elliptic curve whose Weierstrass forms are inherited from the Weierstrass forms of $C \otimes_{F^3} F^3(j)$ and so with the same j-invariant. □

CHAPTER 4

Rational elliptic surfaces with Mordell-Weil rank zero

In this chapter we return to our main interest introduced in the end of Chapter 2, namely the description of the set $\mathscr{H}$ of equivalence classes of absolutely elliptic fibrations by singular curves $f\colon S \to \mathbb{P}^1$ on surfaces S, such that the total spaces S_1 of the Shimada's smoothings f_1 are rational surfaces. Here we are considering the total space and the base of fibrations as varieties over an algebraically closed field k of characteristic three.

The Frobenius morphism $F_{\mathbb{P}^1,k}$ corresponds to the base field extension $k(\mathbb{P}^1)^{1/3} = k(t)^{1/3} = k(T)$ of $k(t)$, where $T^3 = t$. Hence we can consider $K = k(t)$ and $L = k(T)$ in Proposition 3.5 in order to view the generic fibers of the fibrations involved in Shimada's process as the curves studied in Chapter 3.

As we have seen in Proposition 3.12, the study of the generic fibers of fibrations by absolutely elliptic curves of genus two, is equivalent to the study of the generic fibers of their smoothings. In this way, by using Proposition 3.14, we can change our viewpoint from $\mathscr{H}$ to $\mathscr{E}$, the set of equivalence classes of elliptic fibrations (see below) on rational surfaces whose j-invariants of their generic fibers are not cubic powers in $k(T)$.

Now we briefly recall some basic terminology on the theory of rational elliptic surfaces. Let $\mathcal{E}$ be a smooth projective surface over k. We say that $\mathcal{E}$ is a *rational elliptic surface* when $\mathcal{E}$ is a rational surface and there is an *elliptic fibration* $\pi\colon \mathcal{E} \to \mathbb{P}^1$, that is, a fibration such that almost all fibers are elliptic curves and no fiber contains an exceptional curve of first kind. We assume that π has a global section O. For instance, the resolution of base points of a pencil of plane cubic curves gives rise to a rational elliptic surface. Actually, it follows, from [**CD**] Theorem 5.6.1, that any elliptic fibration on rational surfaces is equivalent to a fibration arising in this way.

Remark 4.1. *Notice that each smoothing of absolutely elliptic fibration of genus two possesses a section induced by the rational point mentioned in Proposition 3.5.*

Our strategy to describe $\mathscr{E}$ will use birational invariants of generic fibers of elliptic fibrations. To do this, we need to recall some basic information about elliptic curves.

Let E denote the generic fiber of π, which is an elliptic curve over the function field $k(\mathbb{P}^1) = k(T)$. It can be given by a Weierstrass equation

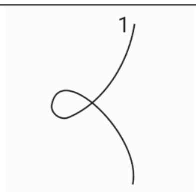
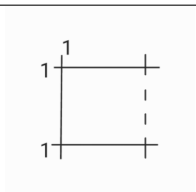
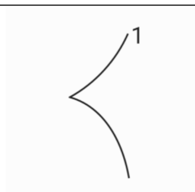
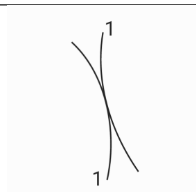
$$y^2 = x^3 + a_2x^2 + a_4x + a_6$$

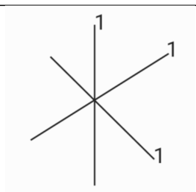
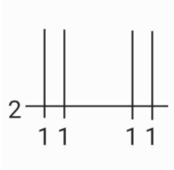
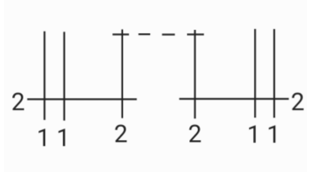
with $a_2, a_4, a_6 \in k[T]$ and $\deg a_i \leq i$ (see **[Shi]** p. 32). In this case the discriminant $\Delta = -a_2^2(a_2a_6 - a_4^2) - a_4^3$ is a polynomial of degree at most 12.

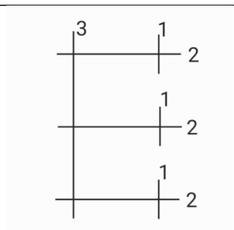
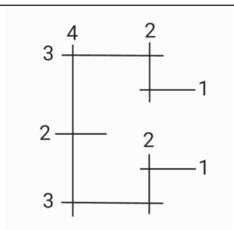
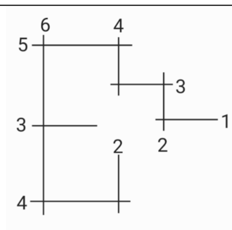
The group $E(k(T))$ of $k(T)$-rational points of E is in a natural one-to-one correspondence with the sections of π and is called the *Mordell-Weil group* of E. From **[Shi]**, Theorem 1.1 and Theorem 10.3, we know that $E(k(T))$ is a finitely generated abelian group with rank

$$\operatorname{rk} E(k(T)) = 8 - \sum_{v \in R} (m_v - 1) \tag{4.1}$$

where $R = \{v \in \mathbb{P}^1 \,|\, F_v := \pi^{-1}(v) \text{ is reducible}\}$ and m_v is the number of components of F_v. The possible special fibers, including the reducible fibers can be seen in the following table.

Kodaira Symbol	I_1	I_m	II	III
Special fiber				

Kodaira Symbol	IV	I_0^*	I_m^*
Special fiber			

Kodaira Symbol	IV^*	III^*	II^*
Special fiber			

Remark 4.2. *Recall that we have the following association among root lattices, special fibers and numbers of its components.*

Root Lattice	A_{m-1} $(m \geq 4)$	D_{m+4}	E_8	E_7	E_6	A_2	A_1
Kodaira Symbol	I_m	I_m^*	II^*	III^*	IV^*	I_3 and IV	I_2 and III
$\sharp$ Components	m	$m+5$	9	8	7	3	2

Table 4.1: Root Lattices, special fibers and numbers of components

We also recall the correspondence below (cf. **[JLRRSP]** *Table* 1*), among a special fiber* $F_v = \pi^{-1}(v)$ *and the order* δ *of* v *as a root of* Δ*.*

Kodaira Symbol	I_m	II	III	I_0^*	I_1^*	I_2^*	I_4^*	III^*	II^*	IV
δ	m	≥ 3	3	6	7	8	10	9	11 or 12	$\geqslant 5$

Table 4.2: Root Orders of Δ

Denoting by T_v the root lattice associated to a reducible fibre F_v. In a elliptic surface the lattice $\mathcal{T} = \bigoplus_{v \in R} T_v$, called trivial lattice, determines the Mordell-Weil group. Indeed, Shioda shows in [**Shi**] that the Mordell-Weil group of an elliptic surface is isomorphic to its Néron-Severi group quotiented by its trivial lattice. In [**OS**] Oguiso and Shioda give all the possible structures for the Mordell-Weil group $E(k(T))$ of a rational elliptic surface.

Since the Mordell-Weil group is invariant under our notion of equivalence of fibrations, we may stratify the set

$$\mathscr{E} = \bigsqcup_{i=0}^{8} \mathscr{E}_i$$

where $\mathscr{E}_i$ is the subset of $\mathscr{E}$ of equivalence classes of elliptic fibrations with Mordell-Weil rank i. In this work we will be concerned with the part $\mathscr{E}_0$.

We state below only the rank zero information of Oguiso-Shioda's theorem.

Theorem 4.3. *The trivial lattice $\mathcal{T}$ and the Mordell-Weil lattice $E(k(T))$ of a rational elliptic surface with Mordell-Weil rank zero are given by the following list.*

1. $\mathcal{T} = E_8$ *and* $E(k(T)) = 0$;
2. $\mathcal{T} = A_8$ *and* $E(k(T)) = \mathbb{Z}/3\mathbb{Z}$;
3. $\mathcal{T} = D_8$ *and* $E(k(T)) = \mathbb{Z}/2\mathbb{Z}$;
4. $\mathcal{T} = E_7 \oplus A_1$ *and* $E(k(T)) = \mathbb{Z}/2\mathbb{Z}$;
5. $\mathcal{T} = A_5 \oplus A_2 \oplus A_1$ *and* $E(k(T)) = \mathbb{Z}/6\mathbb{Z}$;
6. $\mathcal{T} = A_4^{\oplus 2}$ *and* $E(k(T)) = \mathbb{Z}/5\mathbb{Z}$;
7. $\mathcal{T} = A_2^{\oplus 4}$ *and* $E(k(T)) = (\mathbb{Z}/3\mathbb{Z})^2$;
8. $\mathcal{T} = E_6 \oplus A_2$ *and* $E(k(T)) = \mathbb{Z}/3\mathbb{Z}$;
9. $\mathcal{T} = A_7 \oplus A_1$ *and* $E(k(T)) = \mathbb{Z}/4\mathbb{Z}$;
10. $\mathcal{T} = D_6 \oplus A_1^{\oplus 2}$ *and* $E(k(T)) = (\mathbb{Z}/2\mathbb{Z})^2$;
11. $\mathcal{T} = D_5 \oplus A_3$ *and* $E(k(T)) = \mathbb{Z}/4\mathbb{Z}$;
12. $\mathcal{T} = D_4^{\oplus 2}$ *and* $E(k(T)) = (\mathbb{Z}/2\mathbb{Z})^2$;
13. $\mathcal{T} = (A_3 \oplus A_1)^{\oplus 2}$ *and* $E(k(T)) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$;

Using the previous remark we can remove from this list the cases arising from fibrations whose generic fibers have cubic j-invariant. More precisely:

Lemma 4.4. *Over an algebraically closed field k of characteristic 3 we have the following possibilities for special fibers of rational elliptic surfaces with Mordell-Weil rank 0 and j-invariant in $k(T) \setminus k(T)^3$.*

1. II^*, I_1 *when* $\mathcal{T} = E_8$*;*
2. $I_4^*, 2I_1$ *when* $\mathcal{T} = D_8$*;*
3. III^*, I_2, I_1 *when* $\mathcal{T} = E_7 \oplus A_1$*;*
4. $2I_5, 2I_1$ *when* $\mathcal{T} = A_4^{\oplus 2}$*;*
5. $I_8, I_2, 2I_1$ *when* $\mathcal{T} = A_7 \oplus A_1$*;*
6. $I_2^*, 2I_2$ *when* $\mathcal{T} = D_6 \oplus A_1^{\oplus 2}$*;*
7. I_1^*, I_4, I_1 *when* $\mathcal{T} = D_5 \oplus A_3$*;*
8. $2I_4, 2I_2$ *when* $\mathcal{T} = (A_3 \oplus A_1)^{\oplus 2}$.

Proof. To begin with we eliminate the cases of Theorem 4.3 having cubic j-invariant. Notice that the cases 2, 5, 7 and 8 admit at least two points of order dividing 3. On the other hand these points correspond to flex points in a Weierstrass equation. From [**La**] (2.6) pg. 302, an elliptic curve over a field of characteristic 3, with j-invariant different from zero, admits a Weierstrass equation of the form $y^2z = x^3 + a_2x^2z + a_6z^3$ with $\Delta = -a_2^3a_6$ and $j = -a_2^3/a_6$. Hence its flex points are $(0 : 1 : 0)$, $(-a_6^{1/3} : a_2^{1/2} : 1)$ and $(-a_6^{1/3} : -a_2^{1/2} : 1)$, over $\overline{k(T)}$, and we may conclude that elliptic curves E with at least two order three points in $E(k(T))$ must have j-invariant in $k(T)^3$. Indeed, the points $(-a_6^{1/3} : a_2^{1/2} : 1)$ and $(-a_6^{1/3} : -a_2^{1/2} : 1)$ will be the two order three points in $E(k(T))$, so $a_6 \in k(T)^3$. We do not need to consider case 12 either, since each D_4 is associated to a special fiber of type I_0^* and, from the second table of previous remark, we deduce that Δ and consequently j belongs to $k(T)^3$.

For all root lattices appearing in the statement of this lemma, but item 5, if we compare special fibers and their possible orders δ as roots of Δ (summing up to 12), together with (4.1), we obtain, when j is not a cubic power, that the special fibers are necessarily as described. At last, for the case 5 this analysis provides the two possibilities $I_8, I_2, 2I_1$ and I_8, III, I_1 as special fibers when $\mathcal{T} = A_7 \oplus A_1$. However a rational elliptic surface with special fibers I_8, III, I_1 does not exist, from [**JLRRSP**] 4.2.7(28). □

Under the same assumptions as in the previous lemma we have:

Corollary 4.5. *Mordell-Weil rank zero elliptic fibrations with distinct root lattices can not be equivalent.*

Proof. Since the Mordell-Weil group is invariant under birational equivalence we may restrict our attention to the cases with same group. Besides, since automorphisms of $\mathbb{P}^1$ preserve root orders of the discriminant Δ, Lemma 4.4 and Table 4.2 say that fibrations with $\mathcal{T} = D_8$ and $\mathcal{T} = E_7 \oplus A_1$ (respectively $\mathcal{T} = D_5 \oplus A_3$ and $\mathcal{T} = A_7 \oplus A_1$) are not equivalent. □

As a consequence from this corollary

$$\mathscr{E}_0 = \bigsqcup_{\ell=1}^{8} \mathscr{E}_{0,\ell}$$

where $\mathscr{E}_{0,\ell}$ is the subset of $\mathscr{E}_0$ consisting of equivalence classes of elliptic fibrations with lattice as in Lemma 4.4 item ℓ. From now on we will focus our attention to describe each $\mathscr{E}_{0,\ell}$.

We say that a base point $P \in \mathbb{P}^2$ of a pencil Λ of plane curves with no common components is a point with index n when the intersection index at P between any two distinct members of Λ is equal to n.

Theorem 4.6. *Let $\mathcal{E}$ be a rational elliptic surface over an algebraically closed field k of characteristic 3 with Mordell-Weil rank 0, j-invariant in $k(T) \setminus k(T)^3$ and trivial lattice $\mathcal{T}$. Then the elliptic fibration $\pi \colon \mathcal{E} \to \mathbb{P}^1$ is equivalent to a fibration obtained from the resolution of base points of a pencil Λ as described below.*

1. *If $\mathcal{T} = E_8$, then $\Lambda = \langle D, 3L\rangle$, where D is an irreducible nodal cubic curve and L is its inflectional line (cf. Figure 4.1);*

2. *If $\mathcal{T} = D_8$, then $\Lambda = \langle D, L_1 + 2L_2\rangle$ where D is an irreducible nodal cubic curve, L_1 is its inflectional line and L_2 is the line passing through the flex point, tangent to D at a smooth point (cf. Figure 4.2);*

3. *If $\mathcal{T} = E_7 \oplus A_1$, then $\Lambda = \langle D, L_1 + 2L_2\rangle$, where D is an irreducible nodal cubic curve, L_1 is the line through the node and the flex point of D and L_2 is the inflectional line (cf. Figure 4.4);*

4. *If $\mathcal{T} = A_4^{\oplus 2}$, then $\Lambda = \langle D, Q_1 + S_1\rangle$, where D is an irreducible nodal cubic curve, Q_1 is an irreducible conic curve intersecting D at the node and the flex with indices 5 and 1, respectively, and S_1 is the flex line (cf. Figure 4.6);*

5. *If $\mathcal{T} = A_7 \oplus A_1$, then $\Lambda = \langle D, L_1 + L_2 + L_3\rangle$, where D is an irreducible nodal cubic, L_1 is its inflectional line and L_2, L_3 are lines with their intersections represented in Figure 4.10;*

6. *If $\mathcal{T} = D_6 \oplus A_1^{\oplus 2}$, then $\Lambda = \langle D, L_1 + 2L_2\rangle$, where D is an irreducible nodal cubic curve, L_1 is the line through the flex and the node and L_2 is the line through the flex, tangent at a smooth point of D (cf. Figure 4.22);*

7. *If $\mathcal{T} = D_5 \oplus A_3$, then $\Lambda = \langle D, Q_1 + S_1\rangle$, where D is an irreducible nodal cubic curve, Q_1 is an irreducible conic curve intersecting D at the node and the flex with indices 4 and 2, respectively, and S_1 is the flex line (cf. Figure 4.14);*

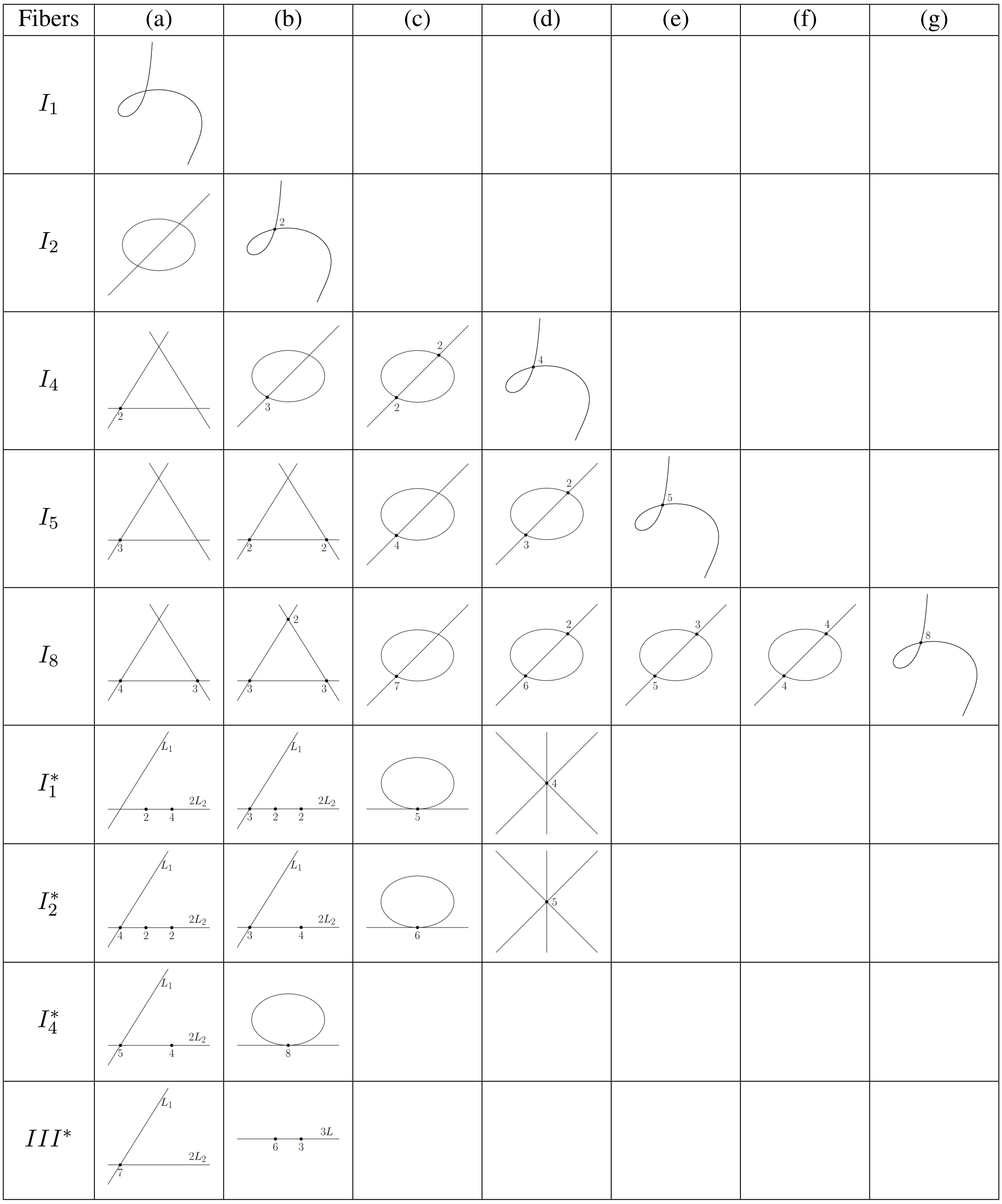
8. *If $\mathcal{T} = (A_3 \oplus A_1)^{\oplus 2}$, then $\Lambda = \langle Q_1 + L_1, Q_2 + L_2\rangle$, where Q_1, Q_2 are irreducible conic curves and L_1, L_2 are lines with their intersections represented in Figure 4.19.*

Proof. We describe the strategy of our proof. To begin with, for each special fiber appearing in Lemma 4.4 we collect all possibilities (C, P, n_P) of singular plane cubic curves C with marked singular points P - and indices n_P attached to them - for which there exists a sequence of $\sum n_P$ blowing-ups at the marked points, whose total transforms provide the given special fiber. (cf. Appendix A)

The second step in our proof will be to select, for each $\mathcal{T}$, the pencils generated by two cubic curves appearing as contractions of the two special fibers with higher number of components. This choice of generators can be made up to an automorphism of $\mathbb{P}^1$ and will restrict the values the Mordell-Weil rank may reach. Yet some of these pencils will give rise to rational elliptic surfaces with rank higher than zero and will be discarded.

The third and final step is to study the equivalence between two fibrations, with same $\mathcal{T}$, obtained from distinct pencils. To this end we will need to identify the group of sections of the elliptic fibrations - or, equivalently, the group $E(k(T))$ - since we will reach the equivalence by blowing-up base points of a pencil and contracting, in a different way, the sections and components of special fibers to obtain the second pencil.

Now we list the possible contractions for each special fiber appearing in Lemma 4.4.



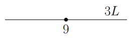
Fibers	(a)	(b)	(c)	(d)	(e)	(f)	(g)
II^*							

Table 4.3: Contractions

In the remaining of the proof we will analyze each case separately. To organize the analysis we will use the notation below.

If $P_r^{(l)}$ is a base point with index n_r of a pencil Λ_l of plane cubic curves, we will denote, when $s < n_r$, by $E_{r,s}^{(l)}$ the exceptional divisor of the s-th blow-up over P_r, and by $\sigma_r^{(l)}$ the exceptional divisor of the n_r-th blow-up over P_r. Also, we refer to the Table 4.3 in the following way: by $(I_5, (d))$ we will mean the contraction of the fiber I_5 illustrated in column (d).

We notice that the exceptional divisors $\sigma_r^{(l)}$ determine disjoint sections of the elliptic fibration or, equivalently, distinct elements in $E(k(T))$. However the number of these $\sigma_r^{(l)}$ is sometimes smaller than the order of $E(k(T))$ and, in this case, we need to find its remaining elements. Also, as a *general principle* used to discard pencils in our analysis, we observe that a base point P of a pencil Λ with smooth general member can be a singular point of at most one member of Λ.

$\boxed{\mathcal{T} = E_8}$ By Lemma 4.4, we have a fiber of type I_1 and a fiber of type II^*. It is immediate to notice from the figures $(II^*, (a))$ and $(I_1, (a))$ that the pencil is generated by $\Lambda_1 = \langle D, 3L\rangle$, where L is the inflectional line of the nodal irreducible cubic D. A configuration of the intersection of generators and resolution of the base point can be seen in Figure 4.1.

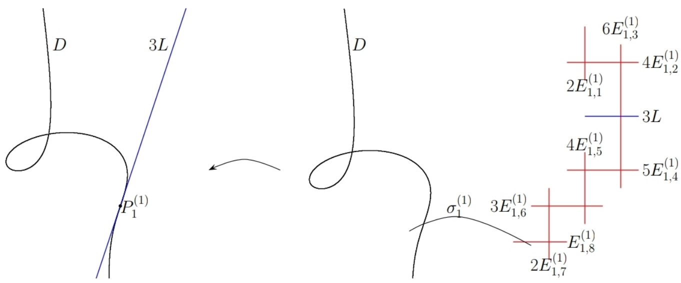


Figure 4.1: Resolution of Λ_1

$\boxed{\mathcal{T} = D_8}$ By Lemma 4.4, we have a fiber of type I_4^* and two fibers of type I_1. We have two pencils $\Lambda_1 = \langle D, L_1 + 2L_2\rangle$ and $\Lambda_2 = \langle D', Q + L\rangle$, where D, D' are like in $(I_1, (a))$ and $L_1 + 2L_2, Q + L$ are like in $(I_4^*, (a))$, $(I_4^*, (b))$, respectively.

In order to determine the geometric configuration of Λ_1 we need to analyze the possibilities of having base points like in $(I_4^*, (a))$. For that it will be enough to analyze if the point $P_1^{(1)}$, as a smooth point of D, is either an inflectional point or not to obtain intersection index 5. It is immediate to see that the only way to get both indices right is if L_1 is the inflectional line to D and L_2 is the line through the flex and tangent to D at another smooth point. For the geometric configuration of Λ_2 we do a similar analysis for the tangency point, $P_1^{(2)}$, between Q and L. The only way to get index 8 at $P_1^{(2)}$ is if it is a non-inflectional point of D' and the irreducible conic Q intersects D' with index 6 and L the tangent line at this point. In figures 4.2 and 4.3 we can see representations of each configuration together with the resolution of the respective base points.

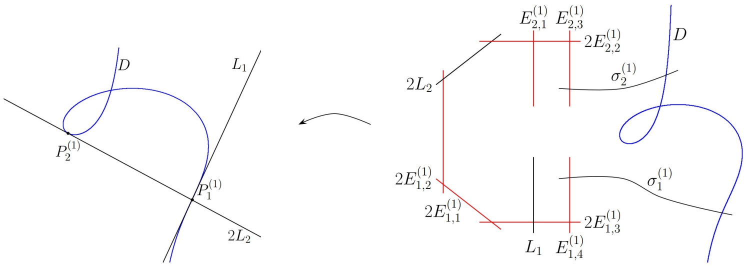


Figure 4.2: Resolution of Λ_1

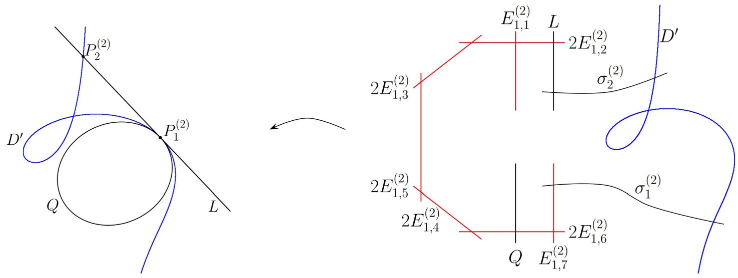


Figure 4.3: Resolution of Λ_2

Now we will show that a fibration produced by a pencil as in Λ_2 is equivalent to a fibration induced by Λ_1. To do this we just need to contract the resolution of Λ_2 as in the sequence

$$\sigma_1^{(2)} \to E_{1,7}^{(2)} \to E_{1,6}^{(2)} \to Q \to \sigma_2^{(2)} \to L \to E_{1,2}^{(2)} \to E_{1,3}^{(2)} \to E_{1,4}^{(2)},$$

to obtain the pencil $\langle D', E_{1,1}^{(2)} + 2E_{1,5}^{(2)} \rangle$ which has same geometric configuration of Λ_1.

$\boxed{\mathcal{T} = E_7 \oplus A_1}$ By Lemma 4.4, we have special fibers of type III^*, I_2 and I_1. Hence the two fibers with higher number of components are III^* and I_2. The combinations $(III^*, (a))$, $(I_2, (a))$ and $(III^*, (b))$, $(I_2, (b))$ can not occur. Indeed, according to the general principle, the singular points of $(III^*, (a))$ and $(I_2, (b))$ must be smooth points of the other curve in each respective pair. However this together with the prescribed indices would provide a contradiction to Bézout's theorem. The remaining combinations give rise to pencils $\Lambda_1 = \langle D, L_1 + 2L_2 \rangle$ and $\Lambda_2 = \langle Q + L, 3L' \rangle$, where $L_1 + 2L_2$, $3L'$, $Q + L$, D are like in $(III^*, (a))$, $(III^*, (b))$, $(I_2, (a))$ and $(I_2, (b))$, respectively. In Λ_1, to obtain index 2 at the node of D, the simple line L_1 must be a line through the node of D which is not in its tangent cone. So the only way of having a base point of index 7 is if L_1 also intersects D at the flex point and L_2 is the inflectional line. In Λ_2 we just need the triple line to be tangent to Q at some smooth point of $Q + L$. By Table 4.2 and the fact that sum of orders as roots of Δ is equal to 12, both pencils must contain a nodal cubic curve as in $(I_1, (a))$. In figures 4.4 and 4.5 we can see representations of each configuration together with the resolution of the respective base points.

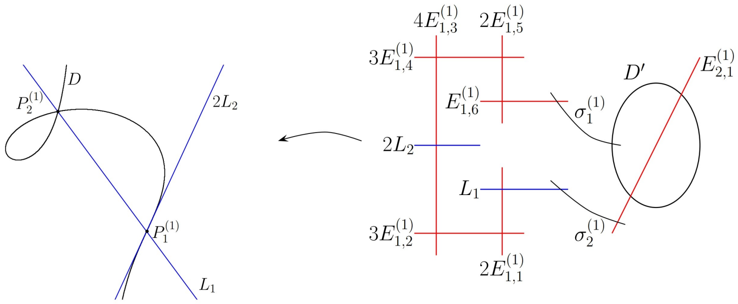


Figure 4.4: Resolution of Λ_1

Now we will show that a fibration produced by a pencil as in Λ_2 is equivalent to a fibration induced by Λ_1. To do this we just need to contract the resolution of Λ_2 as in the sequence

$$\sigma_1^{(2)} \to E_{1,5}^{(2)} \to E_{1,4}^{(2)} \to E_{1,3}^{(2)} \to E_{1,2}^{(2)} \to L' \to E_{2,1}^{(2)} \to \sigma_2^{(2)} \to L,$$

to obtain the pencil $\langle Q, E_{2,2}^{(2)} + 2E_{1,1}^{(2)} \rangle$ which has same geometric configuration of Λ_1, where the curve Q becomes a nodal cubic curve.

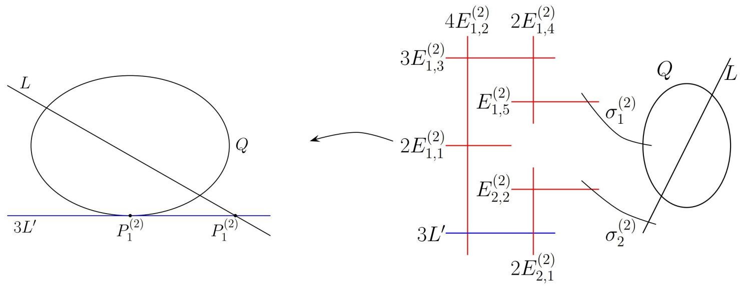


Figure 4.5: Resolution of Λ_2

$\boxed{\mathcal{T} = A_4^{\oplus 2}}$ By Lemma 4.4, we have two special fibers of type I_5 and two of type I_1. The two fibers with higher number of components are the two I_5. We start by analyzing which pairs of contractions of fibers of type I_5 can generate pencils of generically smooth cubic curves. In what follows a pair $((\cdot), (\cdot))$ will mean the pair $((I_5, (\cdot)), (I_5, (\cdot)))$.

$((a), (a))$ Let $L_1 + L_2 + L_3$ and $L'_1 + L'_2 + L'_3$ be the curves and let P_1 and P_2 be the marked points, both having index 3, where P_1 is the intersection point of L_1 and L_2 and P_2 is the intersection point of L'_1 and L'_2. We can not have a pencil with this configuration of curves and indices since we need the marked points to be base points, which can be singular points of only one member of the pencil, by the general principle. However this implies $I_{P_1}(L_1 + L_2 + L_3, L'_1 + L'_2 + L'_3) = I_{P_2}(L_1 + L_2 + L_3, L'_1 + L'_2 + L'_3) = 2$ and we need both intersection indices to be equal to 3;

$((a), (b))$ Let $L_1 + L_2 + L_3$ and $L'_1 + L'_2 + L'_3$ be the curves and let P_1, P_2 and P_3 be the marked points of indices 3, 2 and 2, respectively, where P_1 is the intersection point of L_1 and L_2, P_2 is the intersection point of L'_1 and L'_2 and P_3 is the intersection point of L'_1 and L'_3. For the same reason as the previous case, it is not possible to obtain a geometric configuration in which P_1 is a base point of index 3;

$((a),(c))$ Let $L_1+L_2+L_3$ and $Q+S$ be the curves and let P_1 and P_2 be the marked points of indices 3 and 4 respectively, where P_1 is the intersection point of L_1 and L_2, P_2 is one of the intersection points of Q and S. It is not possible to obtain a pencil in which P_2 is a base point of index 4 since Bézout's theorem implies that $Q+S$ intersect $L_1+L_2+L_3$ at a smooth point with index at most 3;

$((a),(d))$ This case occurs and will be discussed below;

$((a),(e))$ Let $L_1+L_2+L_3$ and D be the curves and let P_1 and P_2 be the marked points of indices 3 and 5 respectively, where P_1 is the intersection point of L_1 and L_2, P_2 is the node of D. It is not possible to obtain a pencil in which P_2 is a base point of index 5 since Bézout's theorem implies that D can intersect $L_1+L_2+L_3$ at a smooth point with index at most 3;

$((b),(b))$ This case occurs and will be discussed below;

$((b),(c))$ Let $L_1+L_2+L_3$ and $Q+S$ be the curves and let P_1, P_2 and P_3 be the marked points of indices $2, 2$ and 4, respectively, where P_1 is the intersection point of L_1 and L_2, P_2 is the intersection point of L_1 and L_3 and P_3 is one of the intersection points of Q and S. It is not possible to obtain a pencil in which P_3 is a base point of index 4 since Bézout's theorem implies that $Q+S$ intersect $L_1+L_2+L_3$ at a smooth point with index at most 3;

$((b),(d))$ Let $L_1+L_2+L_3$ and $Q+S$ be the curves and let P_1, P_2, P_3 and P_4 be the marked points $2, 2, 2$ and 3. Where P_1 is the intersection point of L_1 and L_2, P_2 is the intersection point of L_1 and L_3, and P_3 and P_4 are the intersection points of Q and S. It is not possible to obtain a pencil in which P_1 and P_2 are base points of index 2 and P_4 is a base point of index 3. Indeed, from $I_{P_1}(L_1+L_2+L_3, Q+S) = I_{P_2}(L_1+L_2+L_3, Q+S) = 2$ we have $I_{P_i}(L_j, Q+S) \geq 1$ for $i = 1, 2$ and $j = 1, 2, 3$, so Bézout's theorem implies $I_{P_4}(L_j, Q+S) \leq 2$ for $j = 1, 2, 3$;

$((b),(e))$ Let $L_1+L_2+L_3$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $2, 2$ and 5, respectively, where P_1 is the intersection point of L_1 and L_2, P_2 is the intersection point of L_1 and L_3 and P_3 is the node of D. It is not possible to obtain a pencil in which P_3 is a base point of index 5 since Bézout's theorem implies D can intersect $L_1+L_2+L_3$ at a smooth point with index at most 3;

$((c),(c))$ Let Q_1+S_1 and Q_2+S_2 be the curves and let P_1 and P_2 be the marked points both of index 4. Where P_1 is one of the intersection points of Q_1 and S_1 and P_2 is one of the intersection points of Q_2 and S_2. It is not possible to obtain a pencil in which P_1 and P_2 are base points of index 4. Indeed, on one hand we should have $P_1 \in Q_2 \setminus S_2$, by the general principle, on the other hand we should have $I_{P_1}(Q_1, Q_2) = 3$, by Bézout's theorem. By an analogous argument we also get $I_{P_2}(Q_1, Q_2) = 3$. However these two equalities contradict Bézout's theorem;

$((c),(d))$ This case occurs and will be discussed below;

$((c),(e))$ This case occurs and will be discussed below;

$((d),(d))$ Let Q_1+S_1 and Q_2+S_2 be the curves and let P_1, P_2, P_3 and P_4 be the marked points of indices $2, 3, 2$ and 3, respectively, where P_1 and P_2 are the intersection points of Q_1 and S_1 and P_3 and P_4 are the intersection points of Q_2 and S_2. It is not possible to obtain a pencil having P_1, P_2, P_3

and P_4 as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((d),(e))$ Let $Q+S$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $2, 3$ and 5, respectively, where P_1 and P_2 are the intersection points of Q and S and P_3 is the node of D. It is not possible to obtain a pencil having P_1, P_2 and P_3 as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((e),(e))$ Let D_1 and D_2 be the curves and let P_1 and P_2 be the marked points, both of index 5. Where P_1 is the node of D_1 and P_2 is the node of D_2. It is not possible to obtain a pencil having P_1 and P_2 as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem.

Now we will describe the geometric configurations making the following pencils possible: $\Lambda_1 = \langle D, Q_1+S_1\rangle$, $\Lambda_2 = \langle L_1+L_2+L_3, Q_2+S_2\rangle$, $\Lambda_3 = \langle R_1+R_2+R_3, R'_1+R'_2+R'_3\rangle$, $\Lambda_4 = \langle Q'_1+S'_1, Q'_2+S'_2\rangle$, where $L_1+L_2+L_3$, $R_1+R_2+R_3$, $R'_1+R'_2+R'_3$, Q_1+S_1, $Q'_1+S'_1$, Q_2+S_2, $Q'_2+S'_2$ and D are like in $(I_5,(a))$, $(I_5,(b))$, $(I_5,(b))$, $(I_5,(c))$, $(I_5,(c))$, $(I_5,(d))$, $(I_5,(d))$ and $(I_5,(e))$, respectively. By Table 4.2 and the fact that sum of orders as roots of Δ is equal to 12, these pencils must contain two curves as in $(I_1,(a))$. In Λ_1 the only way of obtaining index 5 at the node $P_1^{(1)}$ of D, is if $P_1^{(1)} \in Q_1 \setminus S_1$ and the only way of obtaining index 4 at a singular point $P_2^{(1)}$ of Q_1+S_1 is if $P_2^{(1)}$ is the flex of D and S_1 is be the inflectional line, since we already have Q_1 transversal to D at $P_2^{(1)}$, by Bézout's theorem. In Λ_2, let $P_1^{(2)}$ be the intersection point of L_1 and L_2, for it to have index 3 we must have $P_1^{(2)} \in Q_2 \setminus S_2$ with Q_2 being tangent to L_2, without loss of generality, and L_1 intersects Q_2 at another point $P_2^{(2)}$. From here it is immediate to notice that we need Q_2 to be tangent to L_3 at a point $P_3^{(2)}$ and S_2 will be the line through $P_2^{(2)}$ and $P_3^{(2)}$. In Λ_3, let $P_1^{(3)}$ and $P_2^{(3)}$ be the points where R_3 meets R_1 and R_2, respectively. Let $P_3^{(3)}$ be another point of R_1 and $P_4^{(3)}$ another point of R_2, then we can take R'_1 as the line through $P_1^{(3)}$ and $P_4^{(3)}$, R'_2 as the line through $P_2^{(3)}$ and $P_3^{(3)}$ and R'_3 as the line through $P_3^{(3)}$ and $P_4^{(3)}$. In Λ_4, let $P_1^{(4)}$ be one of the intersection points of Q'_1 and S'_1, for it to be a base point of index 4 we must have, by the general principle, $P_1^{(4)} \in Q'_2 \setminus S'_2$ and Q'_1 intersects Q'_2 with index 3 at this point. In figures 4.6, 4.7, 4.8 and 4.9 we can see representations of each configuration together with the resolution of the respective base points.

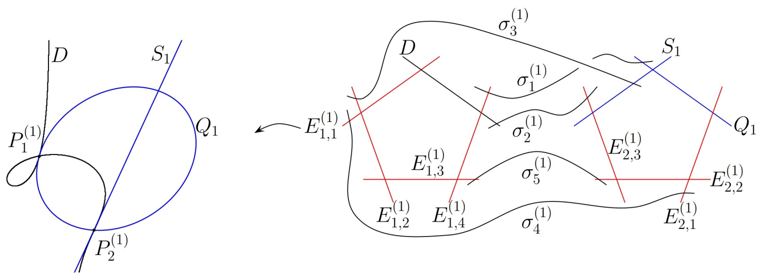


Figure 4.6: Resolution of Λ_1

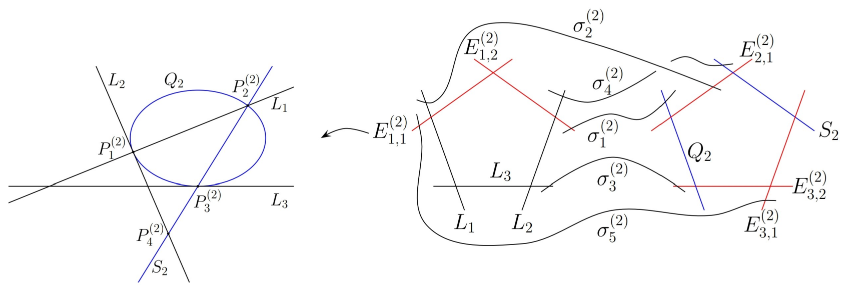


Figure 4.7: Resolution of Λ_2

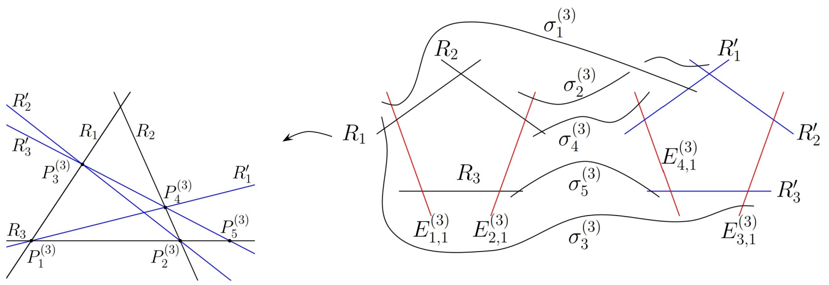


Figure 4.8: Resolution of Λ_3

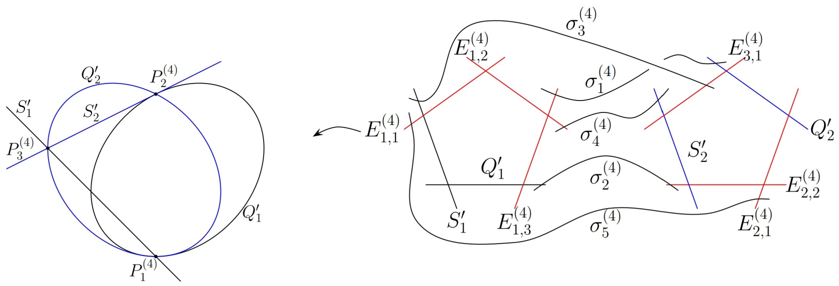


Figure 4.9: Resolution of Λ_4

In the resolution of Λ_1 the sections $\sigma_3^{(1)}$, $\sigma_4^{(1)}$ and $\sigma_5^{(1)}$ are, respectively, the proper transforms of the tangent line to Q_1 at $P_1^{(1)}$, the line through $P_1^{(1)}$ and $P_2^{(1)}$ and the irreducible conic curve intersecting D with index 2 at the flex $P_2^{(1)}$ and intersecting Q_1 with index 3 at the node $P_1^{(1)}$ of D. In the resolution of Λ_2 the section $\sigma_5^{(2)}$ is the proper transform of the line through $P_1^{(2)}$ and $P_3^{(2)}$. In the resolution of Λ_4 the sections $\sigma_4^{(4)}$ and $\sigma_5^{(4)}$ are, respectively, the proper transforms of the tangent line to Q_1 at $P_1^{(4)}$ and the line through $P_1^{(4)}$ and $P_2^{(4)}$.

As before we show that fibrations obtained from Λ_i, $i = 2, 3, 4$, are equivalent to a fibration induced by a pencil Λ_1. We do this by contracting each resolution in an appropriate order.

We contract the resolution of Λ_2 as in the sequence

$$\sigma_4^{(2)} \to L_2 \to L_3 \to L_1 \to E_{1,1}^{(2)} \to \sigma_1^{(2)} \to Q_2 \to E_{3,2}^{(2)} \to E_{3,1}^{(2)}.$$

For Λ_3 we have the sequence

$$\sigma_2^{(3)} \to E_{2,1}^{(3)} \to R_3 \to E_{1,1}^{(3)} \to R_1 \to \sigma_4^{(3)} \to E_{4,1}^{(3)} \to R_3' \to E_{3,1}^{(3)}.$$

For Λ_4 we have the sequence

$$\sigma_1^{(4)} \to E_{1,3}^{(4)} \to Q_1' \to S_1' \to E_{1,1}^{(4)} \to \sigma_4^{(4)} \to S_2' \to E_{2,2}^{(4)} \to E_{2,1}^{(4)}.$$

After these contractions we get the pencils $\langle E^{(2)}_{1,2}, E^{(2)}_{2,1} + S_2\rangle$, $\langle R_2, R'_1 + R'_2\rangle$, $\langle E^{(4)}_{1,2}, E^{(4)}_{3,1} + Q'_2\rangle$, respectively, all with same geometric configuration of Λ_1, where the irreducible nodal cubic curves are the images of $E^{(2)}_{1,2}, R_2, E^{(4)}_{1,2}$ and the irreducible conic curves are the images of $E^{(2)}_{2,1}, R'_1$ and $E^{(4)}_{3,1}$.

$\boxed{\mathcal{T} = A_7 \oplus A_1}$ By Lemma 4.4, we have one special fiber of type I_8, one of type I_2 and two of type I_1. The two fibers with higher number of components are the I_8 and the I_2. We start by analyzing which pairs of contractions of these two special fibers can generate pencils of generically smooth cubic curves having this set of special fibers. In what follows a pair $((\cdot), (\cdot))$ will mean the pair $((I_8, (\cdot)), (I_2, (\cdot)))$.

$((a), (a))$ Let $L_1 + L_2 + L_3$ and $Q + S$ be the curves and let P_1 and P_2 be the marked points of indices 4 and 3, respectively, where P_1 is the intersection point of L_1 and L_2 and P_2 is the intersection point of L_1 and L_3. Since a line can intersect a conic curve with index at most 2, it is not possible to obtain a geometric configuration in which P_1 is a base point of index 4 being singular only in $L_1 + L_2 + L_3$;

$((a), (b))$ This case occurs and will be discussed below;

$((b), (a))$ This case occurs and will be discussed below;

$(((b), (b))$ Let $L_1 + L_2 + L_3$ and D be the curves and let P_1, P_2, P_3 and P_4 be the marked points of indices $3, 3, 2$ and 2, respectively.Where P_1, P_2 and P_3 are the singular points of $L_1 + L_2 + L_3$ and P_4 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((c), (a))$ Let $Q_1 + S_1$ and $Q_2 + S_2$ be the curves and let P_1, one of the intersection points of Q_1 and S_1, be the marked point of index 7. It is not possible to obtain a pencil having P_1 as a base points with the given index, since the general principle requires it to be a smooth point of $Q_2 + S_2$, however this contradicts Bézout's theorem.

$((c), (b))$ This case occurs and will be discussed below;

$((d), (a))$ Let $Q_1 + S_1$ and $Q_2 + S_2$ be the curves and let P_1 and P_2, the intersection points of Q_1 and S_1, be the marked point of indices 6 and 2, respectively. It is not possible to obtain a pencil having P_1 as a base points with the given index, since the general principle requires it to be a smooth point of $Q_2 + S_2$, however this contradicts Bézout's theorem;

$((d), (b))$ Let $Q + S$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $6, 2$ and 2, respectively, where P_1 and P_2 are the singular points of $Q + S$ and P_3 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((e), (a))$ This case occurs and will be discussed below;

$((e), (b))$ Let $Q + S$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $5, 3$ and 2, respectively, where P_1 and P_2 are the singular points of $Q + S$ and P_3 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((f),(a))$ Let $Q+S$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $5, 3$ and 2, respectively, where P_1 and P_2 are the singular points of $Q+S$ and P_3 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would imply $P_1, P_2 \in Q_2 \setminus S_2$ and $I_{P_1}(Q_1, Q_2) = I_{P_2}(Q_1, Q_2) = 3$ which contradicts Bézout's theorem;

$((f),(b))$ Let $Q+S$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices $4, 4$ and 2, respectively, where P_1 and P_2 are the singular points of $Q+S$ and P_3 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((g),(a))$ Let D and $Q+S$ be the curves and let P_1, the node of D, be the marked point of index 8. It is not possible to obtain a pencil having P_1 as a base point with the given index since the general principle requires it to be a smooth point of $Q+S$, however this contradicts Bézout's theorem;

$((g),(b))$ Let D_1 and D_2 be the curves and let P_1 and P_2 be the marked points of indices 8 and 2, respectively, where P_1 is the node of D_1 and P_2 is the node of D_2. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem.

There are other four contractions of a fiber of type I_8 not appearing in the list since they can not be part of a generically smooth pencil. They are:

- The union of an irreducible conic curve and a transversal line with the two intersection points being base points of index 4;
- The union of three lines in general position with one of the intersection points being a base point of index 6;
- The union of three lines in general position with one of the intersection points being a base point of index 5 and another of index 2;
- The union of three lines in general position with one of the intersection points being a base point of index 4 and the other two of index 2.

Now we will describe the geometric configurations making the following pencils possible: $\Lambda_1 = \langle D, L_1 + L_2 + L_3 \rangle$, $\Lambda_2 = \langle Q_1 + S_1, R_1 + R_2 + R_3 \rangle$, $\Lambda_3 = \langle Q_1' + S_1', Q_2 + S_2 \rangle$, $\Lambda_4 = \langle D', Q_3 + S_3 \rangle$, where $L_1 + L_2 + L_3$, $R_1 + R_2 + R_3$, $Q_3 + S_3$, $Q_2 + S_2$, $Q_1 + S_1$, $Q_1' + S_1'$, D and D' are like in $(I_8,(a))$, $(I_8,(b))$, $(I_8,(c))$, $(I_8,(e))$, $(I_2,(a))$, $(I_2,(a))$, $(I_2,(b))$ and $(I_2,(b))$, respectively. By Table 4.2 and the fact that sum of orders as roots of Δ equals 12, these pencils must contain two curves as in $(I_1,(a))$. In Λ_1, the geometric configuration becomes easy to determine after noticing that the only one way of obtaining intersection index 4 at one of the singular points of the curve $L_1 + L_2 + L_3$ - while satisfying the general principle - is if the intersection point is the flex point of the nodal cubic curve and one of the lines, say L_1, is the inflectional line. In Λ_2, the geometric configuration is determined knowing that one can only obtain index 3 at a singular point of the curve $R_1 + R_2 + R_3$ when the conic curve Q_1 is tangent to one of the lines through this singular point. In Λ_3, the geometric configuration is determined knowing that one can only obtain index 5 at a singular point of the curve $Q_2 + S_2$ when both conic curves intersect at this point with index 4. In Λ_4, the geometric configuration is determined knowing that one can only obtain index 7 at a singular point of the curve $Q_3 + S_3$ when the conic curve Q_3 intersects the nodal cubic curve D' with index 6 and this intersection point is not a flex point of D'. The geometric configuration of these pencils and their resolutions can be seen in the figures 4.10, 4.11, 4.12 and 4.13.

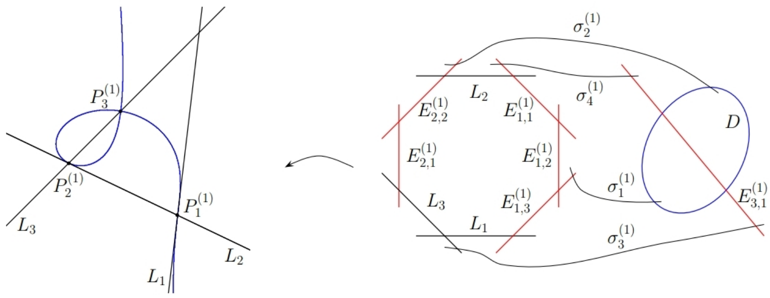


Figure 4.10: Resolution of Λ_1

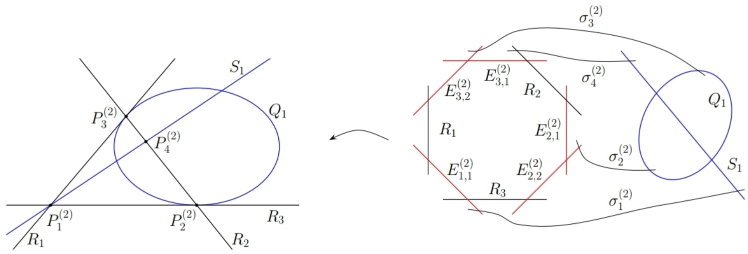


Figure 4.11: Resolution of Λ_2

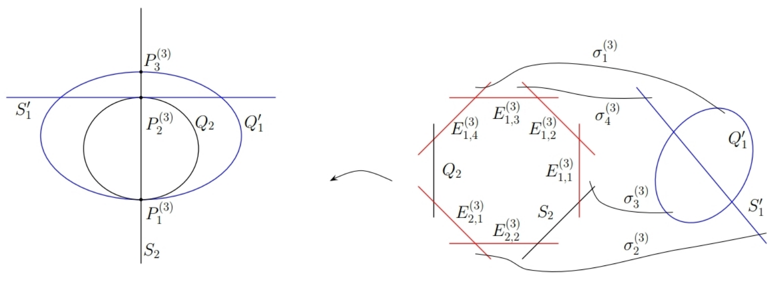


Figure 4.12: Resolution of Λ_3

In the resolution of Λ_1 the section $\sigma_4^{(1)}$ is the proper transform of the line through the flex $P_1^{(1)}$ and the node $P_3^{(1)}$ of D. In the resolution of Λ_3 the section $\sigma_4^{(3)}$ is the proper transform of the tangent line to Q_1 at $P_1^{(3)}$. In the resolution of Λ_4 the sections $\sigma_3^{(4)}$ and $\sigma_4^{(4)}$ are, respectively, the proper transform of the tangent line to D' at $P_1^{(4)}$ and the proper transform of the irreducible conic curve intersecting Q_3 and D' at $P_1^{(4)}$ both with index 3 and intersecting D' at the node $P_2^{(4)}$ with index 2.

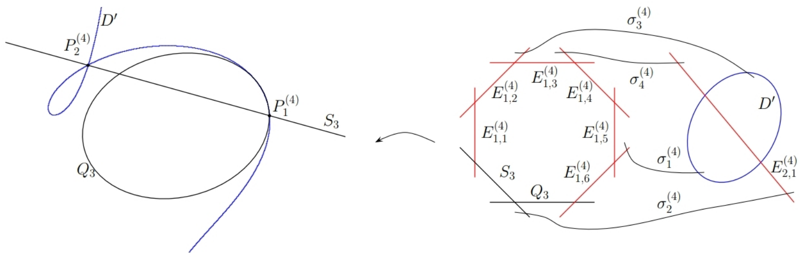


Figure 4.13: Resolution of Λ_4

For Λ_2 we have the sequence

$$\sigma_1^{(2)} \to S_1 \to \sigma_3^{(2)} \to E_{3,2}^{(2)} \to R_1 \to \sigma_2^{(2)} \to E_{2,2}^{(2)} \to E_{2,1}^{(2)} \to R_2.$$

For Λ_3 we have the sequence

$$\sigma_2^{(3)} \to S_1' \to \sigma_1^{(3)} \to E_{1,4}^{(3)} \to Q_2 \to \sigma_3^{(3)} \to S_2 \to E_{1,1}^{(3)} \to E_{1,2}^{(3)}.$$

For Λ_4 we have the sequence

$$\sigma_2^{(4)} \to E_{2,1}^{(4)} \to \sigma_3^{(4)} \to E_{1,2}^{(4)} \to E_{1,1}^{(4)} \to \sigma_1^{(4)} \to E_{1,6}^{(4)} \to E_{1,5}^{(4)} \to E_{1,4}^{(4)}.$$

After these contractions we obtain the pencils $\langle Q_1, E_{1,1}^{(2)} + R_3 + E_{3,1}^{(2)}\rangle$, $\langle Q_1', E_{2,1}^{(3)} + E_{2,2}^{(3)} + E_{1,3}^{(3)}\rangle$, $\langle D', Q_3 + S_3 + E_{1,3}^{(4)}\rangle$, respectively, all with same geometric configuration of Λ_1, the irreducible nodal cubic curves being the images of Q_1, Q_1' and D'.

$\boxed{\mathcal{T} = D_5 \oplus A_3}$ By Lemma 4.4, we have one special fiber of type I_1^*, one of type I_4 and one of type I_1. The two fibers with higher number of components are the I_1^* and the I_4. We start by analyzing which pairs of contractions of fibers of type I_5 can generate pencils of generically smooth cubic curves. In what follows a pair $((\cdot), (\cdot))$ will mean the pair $((I_1^*, (\cdot)), (I_4, (\cdot)))$.

$((a), (a))$ Let $R_1 + 2R_2$ and $L_1 + L_2 + L_3$ be the curves and let P_1, P_2 and P_3 be the marked points of indices 4, 2 and 2, respectively, where P_1 and P_2 are distinct points in R_2 and P_3 is the intersection point of L_1 and L_2. It is not possible to obtain a pencil in which P_1 is a base point of index 4 since $2R_2$ can only intersect $L_1 + L_2 + L_3$ at a smooth point with index 2;

$((a), (b))$ This case occurs and will be discussed below;

$((a), (c))$ Let $R_1 + 2R_2$ and $Q + S$ be the curves and let P_1, P_2, P_3 and P_4 be the marked points of indices 4, 2, 2 and 2, respectively, where P_1 and P_2 are distinct points in R_2 and P_3 and P_4 are the intersection points of Q and S. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((a), (d))$ Let $R_1 + 2R_2$ and D be the curves and let P_1, P_2 and P_3 be the marked points of indices 4, 2 and 4, respectively, where P_1 and P_2 are distinct points in R_2 and P_3 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((b),(a))$ This case occurs and will be discussed below;

$((b),(b))$ Let $R_1 + 2R_2$ and $Q + S$ be the curves and let P_1, P_2, P_3 and P_4 be the marked points of indices 3, 2, 2 and 3, respectively, where P_1 is the intersection point of R_1 and R_2, P_2 and P_3 are distinct points in R_2 and P_4 is one of the intersection points of Q and S. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((b),(c))$ Let $R_1 + 2R_2$ and $Q + S$ be the curves and let P_1, P_2, P_3, P_4 and P_5 be the marked points of indices 3, 2, 2, 2 and 2, respectively, where P_1 is the intersection point of R_1 and R_2, P_2 and P_3 are distinct points in R_2 and P_4 and P_5 are the intersection points of Q and S. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((b),(d))$ Let $R_1 + 2R_2$ and D be the curves and let P_1, P_2, P_3 and P_4 be the marked points of indices 3, 2, 2 and 4, respectively, where P_1 is the intersection point of R_1 and R_2, P_2 and P_3 are distinct points in R_2 and P_4 is the node of D. It is not possible to obtain a pencil having these points as base points with the given indices, since the general principle would make the sum of intersection indices be at least 10 which contradicts Bézout's theorem;

$((c),(a))$ Let $Q + S$ and $L_1 + L_2 + L_3$ be the curves and let P_1 and P_2 be the marked points of indices 5 and 2, respectively, where P_1 is the intersection point of Q and S and P_2 is the intersection point of L_1 and L_2. It is not possible to obtain a pencil in which P_1 is a base point of index 5 since $Q + S$ can only intersect $L_1 + L_2 + L_3$ at a smooth point with index at most 2 (S can not be a component of $L_1 + L_2 + L_3$);

$((c),(b))$ This case occurs and will be discussed below;

$((c),(c))$ Let $Q_1 + S_1$ and $Q_2 + S_2$ be the curves and let P_1, P_2 and P_3 be the marked points of indices 5, 2 and 2, respectively, where P_1 is the intersection point of Q and S and P_2 and P_3 are the intersection points of Q_2 and S_2. It is not possible to obtain a pencil having these points as base points with the given indices, since, on one side, to obtain index 5 at P_1 we need $P_1 \in Q_2 \setminus S_2$, on the other side, P_2 and P_3 are also in Q_2. This means that we should have the sum of intersection indices between $Q_1 + S_1$ and Q_2 being at least 7 which contradicts Bézout's theorem;

$((c),(d))$ This case occurs and will be discussed below;

$((d),(a))$ Let $R_1 + R_2 + R_3$ and $L_1 + L_2 + L_3$ be the curves and let P_1 and P_2 be the marked points of indices 4 and 2, respectively, where P_1 is the common intersection point of R_1, R_2 and R_3 and P_2 is the intersection point of L_1 and L_2. It is not possible to obtain a pencil in which P_1 is a base point of index 4 since $R_1 + R_2 + R_3$ can only intersect $L_1 + L_2 + L_3$ at a smooth point with index at most 3 (R_i can not be a component of $L_1 + L_2 + L_3$, for $i = 1, 2, 3$);

$((d),(b))$ Let $R_1 + R_2 + R_3$ and $Q + S$ be the curves and let P_1 and P_2 be the marked points of indices 4 and 3, respectively, where P_1 is the common intersection point of R_1, R_2 and R_3 and P_2 is one of the intersection point of Q and S. It is not possible to obtain a pencil having these points as base points with the given indices, since to obtain index 4 at $P_1 \in Q \setminus S$ where Q is tangent to one of the three lines, say R_1, which implies we can not obtain index 3 at P_2 since both R_2 and R_3 would be transversal to Q;

$((d),(c))$ This case occurs and will be discussed below;

$((d),(d))$ Let $R_1+R_2+R_3$ and D be the curves and let P_1 and P_2 be the marked points of indices 4 and 4, respectively, where P_1 is the common intersection point of R_1, R_2 and R_3 and P_2 is the node of D. It is not possible to obtain a pencil in which P_2 is a base point of index 4 since D can only intersect $R_1+R_2+R_3$ at a smooth point with index at most 3.

Now we will describe the geometric configurations making the following pencils possible: $\Lambda_1 = \langle D, Q_1+S_1\rangle$, $\Lambda_2 = \langle L_1+L_2+L_3, R_1+2R_2\rangle$, $\Lambda_3 = \langle Q_2+S_2, R_3+2R_4\rangle$, $\Lambda_4 = \langle Q_2'+S_2', Q_1'+S_1'\rangle$, $\Lambda_5 = \langle Q_3+S_3, L_4+L_5+L_6\rangle$, where R_3+2R_4, R_1+2R_2, Q_1+S_1, $Q_1'+S_1'$, $L_4+L_5+L_6$, $L_1+L_2+L_3$, Q_2+S_2, $Q_2'+S_2'$, Q_3+S_3 and D are like in $(I_1^*,(a))$, $(I_1^*,(b))$, $(I_1^*,(c))$, $(I_1^*,(c))$, $(I_1^*,(d))$, $(I_4,(a))$, $(I_4,(b))$, $(I_4,(b))$, $(I_4,(c))$ and $(I_4,(d))$, respectively. By Table 4.2 and since the sum of orders as roots of Δ equals 12, all these pencils must contain a curve as in $(I_1,(a))$. In Λ_1 the geometric configuration becomes easy to determine after noticing that the node of D must be in $Q_1 \setminus S_1$ and to obtain index 5 at the intersection point of Q_1 and S_1, it needs to be the flex point of D and S_1 is its inflectional line. In Λ_2, the intersection point of L_1 and L_2 is a point of R_1, the intersection point of each of these lines with R_2 is a base point of index 2 and L_3 is a third line through the intersection point of R_1 and R_2. In Λ_3 the geometric configuration becomes easy to determine after noticing that the only way of obtaining index 4 at a point of $R_4 \setminus R_3$ is if Q_2 is tangent R_4 at this point, and to obtain index 3 at one of the intersection points of Q_2 and S_2 we need Q_2 to be tangent to R_3 at this point. In Λ_4 the geometric configuration is determined knowing that one can only obtain index 5 at the singular point of the curve $Q_1'+S_1'$ when both conic curves intersect at this point with index 3. In Λ_5 to obtain index 4 at the singular point of the curve $L_4+L_5+L_6$ when the conic curve Q_3 is tangent to one of the lines at this point and the S_3 is the line through the intersection points of Q_3 with the components of $L_4+L_5+L_6$. The geometric configurations of these pencils and the resolution of their base points can be seen in the figures 4.14, 4.15, 4.16, 4.17 and 4.18.

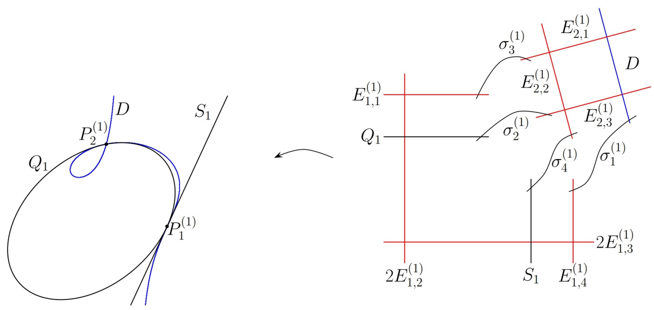


Figure 4.14: Resolution of Λ_1

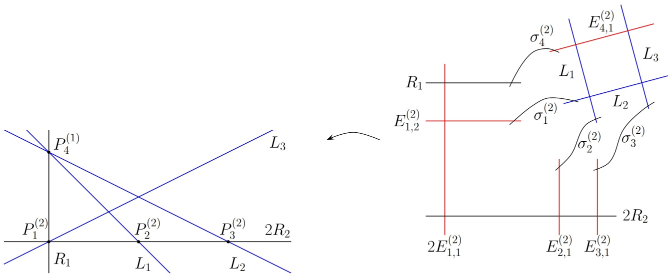


Figure 4.15: Resolution of Λ_2

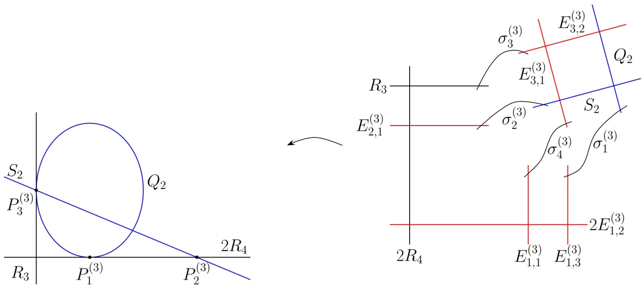


Figure 4.16: Resolution of Λ_3

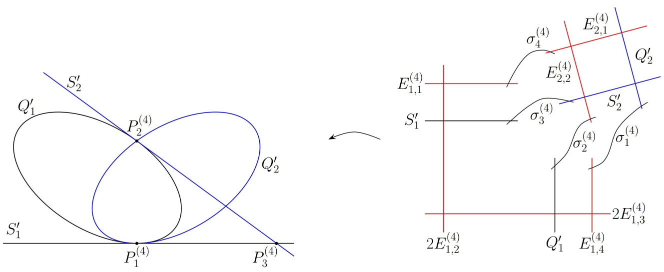


Figure 4.17: Resolution of Λ_4

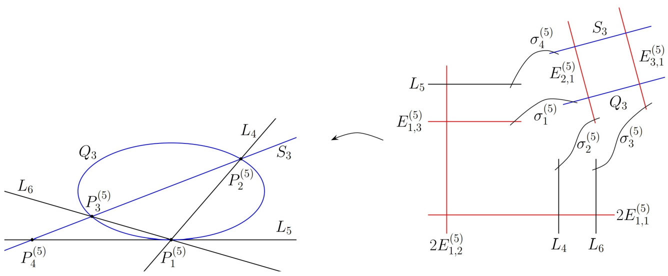


Figure 4.18: Resolution of Λ_5

In the resolution of Λ_1 the sections $\sigma_3^{(1)}$ and $\sigma_4^{(1)}$ are, respectively, the proper transform of the line through the flex $P_1^{(1)}$ and the node $P_2^{(1)}$ of D and the proper transform of the tangent line to Q_1 at $P_2^{(1)}$. In the resolution of Λ_3 the section $\sigma_4^{(3)}$ is the proper transform of the line through $P_1^{(3)}$ and $P_3^{(3)}$. In the resolution of Λ_4 the section $\sigma_4^{(4)}$ is the proper transform of the line through $P_1^{(4)}$ and $P_2^{(4)}$.

For Λ_2 we have the sequence

$$\sigma_1^{(2)} \to L_2 \to L_1 \to E_{4,1}^{(2)} \to \sigma_3^{(2)} \to E_{3,1}^{(2)} \to R_2 \to E_{1,1}^{(2)} \to R_1.$$

For Λ_3 we have the sequence

$$\sigma_2^{(3)} \to S_2 \to E_{3,1}^{(3)} \to E_{3,2}^{(3)} \to \sigma_1^{(3)} \to E_{1,3}^{(3)} \to E_{1,2}^{(3)} \to R_4 \to R_3.$$

For Λ_4 we have the sequence

$$\sigma_3^{(4)} \to S_2' \to E_{2,2}^{(4)} \to E_{2,1}^{(4)} \to \sigma_1^{(4)} \to E_{1,4}^{(4)} \to E_{1,3}^{(4)} \to E_{1,2}^{(4)} \to E_{1,1}^{(4)}.$$

For Λ_5 we have the sequence

$$\sigma_1^{(5)} \to Q_3 \to E_{2,1}^{(5)} \to S_3 \to \sigma_3^{(5)} \to L_6 \to E_{1,1}^{(5)} \to E_{1,2}^{(5)} \to L_5.$$

After these contractions we obtain the pencils $\langle L_3, E_{2,1}^{(2)} + E_{1,2}^{(2)}\rangle$, $\langle Q_2, E_{1,1}^{(3)} + E_{2,1}^{(3)}\rangle$, $\langle Q_2', Q_1' + S_1'\rangle$, $\langle E_{3,1}^{(5)}, L_4 + E_{1,3}^{(5)}\rangle$, respectively, all with same geometric configuration of Λ_1, where the irreducible nodal cubic curves are the images of L_3, Q_2, Q_2' and $E_{3,1}^{(5)}$ and the irreducible conic curve are the images of $E_{2,1}^{(2)}, E_{1,1}^{(3)}, Q_1'$ and L_4.

$\boxed{\mathcal{T} = (A_3 \oplus A_1)^{\oplus 2}}$ By Lemma 4.4, we have two special fibers of type I_4 and two of type I_2. Hence the two fibers with higher number of components are the two I_4. Differently from all cases above, almost every pair of contractions of these two special fibers will yield a pencil, some of the pairs will have multiple ways of intersecting which imply multiple pencils having two special fibers of type I_4. However, most of them will yield elliptic surfaces with a set special fibers different from the one we are looking for. Indeed, the two fibers of type I_4 only indicate two roots of Δ of order 4, that is, we have a degree 4 factor of Δ which we do not know its factorization. We need to impose the existence of certain cubic curves in the pencil, corresponding to the fibers of type I_2. We do this by choosing generators G_1, G_2 with a specific geometric configuration so that there are cubic curves C, not necessarily irreducible, satisfying $I_P(C, G_1) = I_P(C, G_2) = I_P(G_1, G_2)$ for all $P \in \mathbb{P}^2$. By Noether's Fundamental Theorem (see **[F]**, Section 5.5) we guarantee that these curves are members of the pencil.

We will list every configuration in which each pair of generators G_1, G_2 can intersect. In what follows a pair $((\cdot), (\cdot))$ will mean the pair $((I_4, (\cdot)), (I_4, (\cdot)))$.

$((a), (a))$ In this case we have one geometric configuration giving a pencil generated by curves $L_1 + L_2 + L_3$ and $L_1' + L_2' + L_3'$. Let P_1 be the intersection point of L_1 and L_2 and P_2 be the intersection point of L_1' and L_2'. The geometric configuration is determined, up to reordering the components, by putting P_1 in L_2' and P_2 in L_2 and every remaining component should intersect the other curve in three distinct points. We can not have any contraction of a fiber of type I_2 in this pencil. Indeed, since every line through three base points is one of the components of the generators which implies we can not have a contraction as in $(I_2, (a))$, on the other hand each base point of index 2 is a singular point of a generator which implies we can not have a contraction as in $(I_2, (b))$ either.

$((a), (b))$ In this case we have multiple geometric configurations giving pencils generated by curves $L_1 + L_2 + L_3$ and $Q + S$, depending on how they intersect. We must have the intersection point P_1 of L_1 and L_2, without loss of generality, as the base point of index 2 and P_2, one of the intersection points of Q and S as the base point of index 3. We have only one way of obtaining index 3 in a smooth point of $L_1 + L_2 + L_3$, that is, when Q is tangent to L_3. On the other hand, we have two ways of obtaining intersection index 2 at a smooth point of $Q + S$, either $P_1 \in Q \setminus S$ with Q being transversal to both L_1 and L_2 or $P_1 \in S \setminus Q$.

$P_1 \in Q \setminus S$ In this case we have a pencil that can not contain any contraction of a fiber of type I_2 since any line through three base points will be a component of the generators and the only base point of index two is a singular point of a generator.

$P \in S \setminus Q$ This case will still have multiple configurations depending on how Q intersects the components L_1 and L_2 of the first generator.

1. Suppose Q is not tangent to L_1 nor L_2. In this case we have a pencil that can not contain any contraction of a fiber of type I_2 since any line through three base points will be a component of the generators and the only point of index 2 is a singular point of a generator.
2. Suppose Q is tangent to only one the components, say L_1. In this case the pencil may contain only one contraction of a fiber of type I_2 as in $(I_2, (b))$ since we have a base point of index 2 being smooth on both generators.
3. Suppose Q is tangent to both L_1 and L_2 at points P_3 and P_4, respectively. In this case the pencil will contain two contraction of a fiber of type I_2 as in $(I_2, (b))$ since we have two points of index 2 being smooth on both generators. The special fibers over these order two points will be of type I_2 because, on one hand, they can only add up to order 4 in the factorization of Δ, on the other hand, the Lemmas D and G in [**JLRRSP**] imply that in characteristic three a rational elliptic fibration having two fibers of type I_4 must have the remaining fibers of type I_n with $n < 3$.

$(((a), (c))$ In this case we have multiple geometric configurations giving pencils generated by curves $L_1 + L_2 + L_3$ and $Q + S$. Let P_1 be the intersection point of L_1 and L_2 and P_2 and P_3 be the intersection points of Q and S. For P_1 to be a base point of index 2 we must have $P_1 \in Q$, since we already have $P_2, P_3 \in S$. On the other hand, for P_2 and P_3 to be base points of index 2 each we have two possibilities. Firstly we can assume, without loss of generality, $P_2 \in L_1$ then either $P_3 \in L_2$ or $P_3 \in L_3$.

$(P_3 \in L_2)$ Here we have another two possibilities depending on how Q intersects L_3.

1. Suppose Q is not tangent to L_3. In this case we have a pencil that may contain only one contraction of a fiber of type I_2 if the tangent line to Q at P_1 intersects L_3 at the same point as S then there will be an irreducible conic tangent to L_1 at P_2 and tangent to L_3 at P_3. But since every other line through three base points is component of one generator and every base point of index 2 is a singular point of a generator, we can not have another contraction of a fiber of type I_2.
2. Suppose Q is tangent to L_3 at a point P_4. In this case we have a pencil that may contain two contractions of a fiber of type I_2. One of those will be a nodal cubic curve as in $(I_2, (b))$ with node at P_4 and the second one will be as in $(I_2, (a))$ if tangent line to Q at P_1 intersects L_3 at the base point P_5, the intersection point of L_3 and S.

$(P_3 \in L_3)$ In this case we have a pencil that can not contain any contraction of a fiber of type I_2 since any line through three base points will either be a component of the generators or intersect each generator with a different index at one of its points and the only base point of index 2 is a singular point of a generator;

$((a),(d))$ Let $L_1 + L_2 + L_3$ and D be the curves and let P_1 and P_2 be the marked points of indices 2 and 4, respectively, where P_1 is the intersection point of L_1 and L_2 and P_2 is the node of D. It is not possible to obtain a pencil having P_2 as a base point with the given index since the general principle requires it to be a smooth point of $L_1 + L_2 + L_3$, however this contradicts Bézout's theorem;

$((b),(b))$ In this case we have three geometric configurations giving pencils generated by curves $Q_1 + S_1$ and $Q_2 + S_2$. Let P_1 (respectively P_2) be one of the intersection points between Q_1 and S_1 (respectively Q_2 and S_2). In order to obtain index 3 at P_1 (respectively at P_2) we have two possibilities, either Q_1 (resp. Q_2) is tangent to S_2 (resp. S_1) or tangent to Q_2 (resp. Q_1) at P_1 (resp. P_2). Combining these possibilities we obtain the three configurations. It is immediate to see that none of them can contain a contraction of a fiber of type I_2, since there are no base points of index 2 on the pencils and the only line intersecting both curves just on base points which is not one of their components is $\overline{P_1P_2}$, but since this line intersects each generator with a different index it can not be a component of a member of the pencil.

$((b),(c))$ In this case we have multiple geometric configurations giving pencils generated by curves $Q_1 + S_1$ and $Q_2 + S_2$. Let P_1 be one of the intersection points between Q_1 and S_1 and P_2 and P_3 be the intersection points between Q_2 and S_2. In order to obtain index 3 at P_1 we have two possibilities, either Q_1 is tangent to S_2 or Q_2 at this point. On the other hand, there is only one way of obtaining intersection indices 2 at P_2 and P_3, that is, Q_2 is transversal to Q_1 at P_2 and P_3.

1. Suppose Q_1 is tangent to S_2 at P_1, then Q_1 and Q_2 will intersect at four distinct points P_1, P_2, P_3 and P_4. In this case the pencil will not contain any contraction of a fiber of type I_2 since the only base points of index 2 are P_2 and P_3, that are singular points of $Q_2 + S_2$ and the only lines intersecting both generators just at base points are $\overline{P_1P_2}$ and $\overline{P_1P_3}$ but they intersect each generator with a different index, so it can not be a component of any member of the pencil.
2. Suppose Q_1 is tangent to Q_2 at P_1. In this case the pencil may contain two contractions of a fiber of type I_2, both as in $(I_2,(a))$, if the both tangent lines to Q_1 at P_2 and P_3 intersect Q_2 at the base point P_4, the intersection point of Q_2 and S_1. For the tangent line at P_2 we have the irreducible conic curve tangent to Q_1 at P_1 and P_3 and going through the intersection point P_5 of S_1 and S_2, while the conic curve for the other tangent line is obtained in the same way by taking P_2 instead of P_3.

$((b),(d))$ In this case we have two geometric configurations giving pencils generated by curves $Q + S$ and D. Let P_1 be one of the intersection points between Q and S and P_2 be the node of D. For P_2 to be a base point of index 4 we must have $P_2 \in Q \setminus S$. To obtain index 3 at P_1 we have two possibilities, either Q or S intersect D with index 2 at P_1. Firstly notice that can not contain a member as in $(I_2,(b))$ since they do not have a base point of index 2.

1. Suppose D is tangent to Q at P_1. In this case the pencil may contain only one member as in $(I_2,(a))$ since the only way of obtaining a line different from S through base points intersecting each curve with the same index is if P_1 is the flex of D, so the member would be formed by the inflectional line and the irreducible conic intersecting D and Q with index 4 at P_1 and intersecting D at the other two intersection points of D and S.

2. Suppose D is tangent to S at P_1. In this case the pencil can not contain any member as in $(I_2, (a))$ since there is no line through three base points intersecting both generators with the same index.

$((c),(c))$ In this case we have only one geometric configuration giving a pencil generated by curves $Q_1 + S_1$ and Q_2+S_2. Let P_1 and P_2 be the singular points of Q_1+S_1 and let P_3 and P_4 be the singular points of $Q_2 + S_2$. Since all four points must be base points of index 2 we must have $P_1, P_2 \in Q_2 \setminus S_2$ and $P_3, P_4 \in Q_1 \setminus S_1$. However, with this configuration, we can not have any contraction of a fiber of type I_2 since each base point of index 2 is a singular point of a generator and there is no line through base points intersecting both generators with the same index.

$((c),(d))$ In this case we have only one geometric configuration giving a pencil generated by curves $Q + S$ and D. Let P_1 and P_2 be the singular points of $Q + S$ and let P_3 be the node of D. For P_3 to be a base point of index 4, we must have $P_3 \in Q \setminus S$. To obtain indices 2 at P_1 and P_2 the line S must be transversal to D. With this configuration, we can not have any contraction of a fiber of type I_2 since each base point of index 2 is a singular point of a generator and there is no line through base points intersecting both generators with the same index.

$((d),(d))$ In this case we can have only one pencil generated by curves D_1 and D_2. Let P_1 be the node of D_1 and let P_2 be the node of D_2. Since P_1 and P_2 must have index 4, we can not have any contraction of a fiber of type I_2 because, on one side, we do not have any index 2 base point and, on the other side, we can not have two index 4 base points being smooth points of an irreducible conic curve.

Now we will highlight the pencils that we are interested in assuming the conditions that allow the two special fibers of type I_2 in their resolutions and we will relabel the generators and the base points according to the pencil they are in for further use.

- $\Lambda_1 = \langle Q_1 + S_1, Q_2 + S_2\rangle$, where the generators are as in item 2 of the case $((b),(c))$, and we assume the tangent lines to Q_1 at the base points $P_2^{(1)}$ and $P_3^{(1)}$ intersecting Q_2 at the same point as S_1. We will call S_3 the tangent line to Q_1 at $P_2^{(1)}$, Q_3 the conic curve tangent to Q_1 at $P_1^{(1)}$ and $P_3^{(1)}$, S_4 the tangent line to Q_1 at $P_3^{(1)}$ and Q_4 the conic curve tangent to Q_1 at $P_1^{(1)}$ and $P_2^{(1)}$.

- $\Lambda_2 = \langle R_1 + R_2 + R_3, Q + S\rangle$, where the generators are as in item 2 of the case $((a),(c))$, and we assume the tangent line, S_5, to Q at $P_1^{(2)}$ intersects R_3 at the same point as S. We will call Q_5 the conic curve tangent to R_1, R_2 and R_3 at $P_4^{(2)}, P_3^{(2)}$ and $P_2^{(2)}$, respectively, and D the nodal cubic curve with node at $P_4^{(2)}$.

- $\Lambda_3 = \langle L_1 + L_2 + L_3, Q_6 + S_6\rangle$, where the generators are as in item 3 of the case $((a),(b))$. We call D_1 the nodal cubic curve with node at $P_3^{(3)}$ and D_2 the nodal cubic curve with node at $P_4^{(3)}$.

The geometric configurations of these pencils and the resolution of their base points can be seen in the figures 4.19, 4.20 and 4.21.

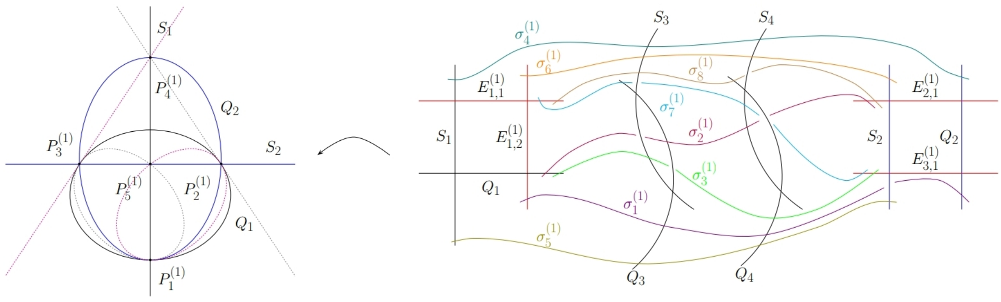


Figure 4.19: Resolution of Λ_1

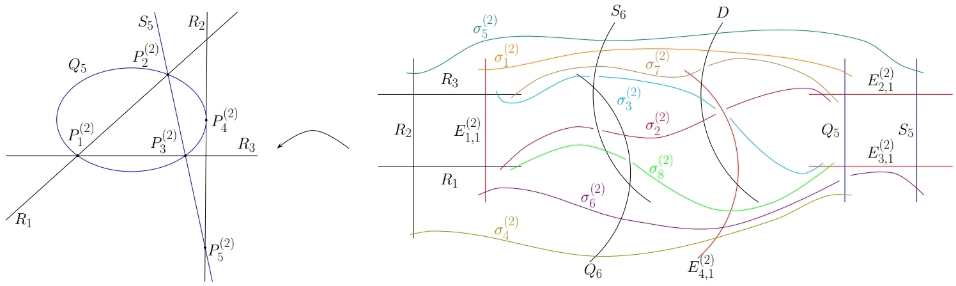


Figure 4.20: Resolution of Λ_2

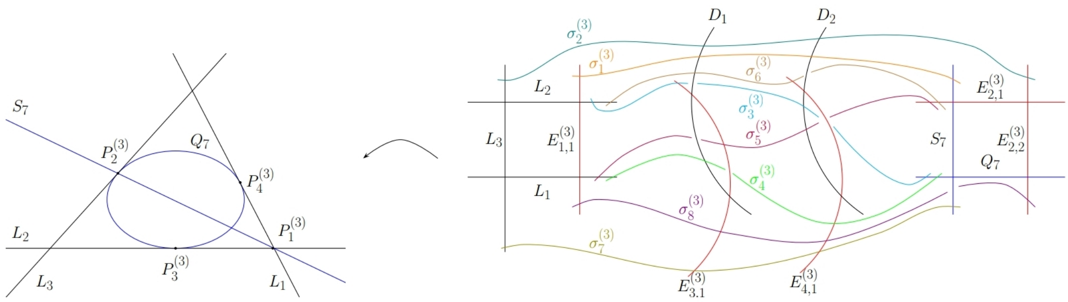


Figure 4.21: Resolution of Λ_3

In the resolution of Λ_1 the sections $\sigma_6^{(1)}, \sigma_7^{(1)}$ and $\sigma_8^{(1)}$ are, respectively, the proper transforms of the tangent line to Q_1 at $P_1^{(1)}$, the line through $P_1^{(1)}$ and $P_2^{(1)}$ and the line through $P_1^{(1)}$ and $P_3^{(1)}$. In the resolution of Λ_2 the sections $\sigma_6^{(2)}, \sigma_7^{(2)}$ and $\sigma_8^{(2)}$ are, respectively, the proper transforms of the line through $P_1^{(2)}$ and $P_4^{(2)}$, the line through $P_2^{(2)}$ and $P_4^{(2)}$ and the line through $P_3^{(2)}$ and $P_4^{(2)}$. In the resolution of Λ_3 the sections $\sigma_5^{(3)}, \sigma_6^{(3)}, \sigma_7^{(3)}$ and $\sigma_8^{(3)}$ are, respectively, the proper transforms of the line through $P_2^{(3)}$ and $P_3^{(3)}$, the line through $P_2^{(3)}$ and $P_4^{(3)}$, the line through $P_3^{(3)}$ and $P_4^{(3)}$ and the irreducible conic curve through all base points, tangent to R_3 at $P_2^{(3)}$.

Now we can follow the sequences below to find pencils with the same configuration of Λ_1.

For Λ_2 we have the sequence

$$\sigma_6^{(2)} \to E_{1,1}^{(2)} \to R_3 \to \sigma_8^{(2)} \to E_{3,1}^{(2)} \to \sigma_2^{(2)} \to E_{2,1}^{(2)} \to \sigma_4^{(2)} \to \sigma_5^{(2)}.$$

For Λ_3 we have the sequence

$$\sigma_8^{(3)} \to E_{1,1}^{(3)} \to L_2 \to \sigma_4^{(3)} \to Q_7 \to \sigma_5^{(3)} \to E_{2,1}^{(3)} \to \sigma_2^{(3)} \to \sigma_6^{(3)}.$$

After these contractions we obtain the pencils $\langle R_1 + R_2, Q_5 + S_5\rangle$ and $\langle L_1 + L_3, S_7 + E_{2,2}^{(3)}\rangle$, respectively, both having the same geometric configuration of Λ_1. The pairs of irreducible conic curves are, respectively, the images of R_1, S_5 and $L_1, E_{2,2}^{(3)}$.

$\boxed{\mathcal{T} = D_6 \oplus A_1^{\oplus 2}}$ By Lemma 4.4, we have one special fiber of type I_2^* and two of type I_2. Hence the two fibers with higher number of components are the I_2^* and one of the I_2. As in the previous case, we have some pairs giving rise to pencils that yield elliptic surfaces with another set of special fibers. But in this case it is simpler to identify the ones we want, since, given a such pencil, we just have some freedom on a factor of degree 2 of Δ which can only give us another fiber of I_2 or two of type I_1.

To find the pencils and conditions needed to obtain the second contraction of a fiber of type I_2 as one of its members, we will list every configuration in which a contraction of a fiber of type I_2^* can intersect a contraction of a fiber of type I_2. In what follows a pair $((\cdot), (\cdot))$ will mean the pair $((I_2^*, (\cdot)), (I_2, (\cdot)))$.

$((a), (a))$ In this case we have a geometric configuration giving a pencil generated by curves $L_1 + 2L_2$ and $Q + S$, where the intersection point P_1 of L_1 and L_2 is a base point of index 4. Notice that the only way of obtaining index 4 at P_1 while being a smooth point of $Q + S$ is if the $P_1 \in Q \setminus S$ and L_1 is the tangent line to Q at this point. Let P_2 be the second intersection point of Q and L_2 and let P_3 and P_4 be the points where S intersects L_1 and L_2, respectively;

$((a), (b))$ In this case we have curves $L_1 + 2L_2$ and D where the intersection point P_1 of L_1 and L_2 should be a base point of index 4 and the node P_2 of D should be a base point of index 2. To obtain a pencil having these base points and giving rise to an elliptic surface, on one side, we would have $P_2 \in L_1 \setminus L_2$ and, on the other side, we would have D tangent to L_1 at P_1, but this contradicts Bézout's theorem;

$((b), (a))$ In this case we have a geometric configuration giving a pencil generated by curves $L_1 + 2L_2$ and $Q + S$, where the intersection point P_1 of L_1 and L_2 is a base point of index 3 and we have another point $P_2 \in L_2 \setminus L_1$ of index 4. The only way these indices can be achieved, while satisfying the general principle, is if Q is tangent to L_2 at P_2 and S is a line through P_1 different from L_1 and L_2;

$((b), (b))$ In this case we have a geometric configuration giving a pencil generated by curves $L_1 + 2L_2$ and D, where the intersection point P_1 of L_1 and L_2 is a base point of index 3, the node P_2 of D is a base point of index 2 and we have another base point $P_3 \in L_2 \setminus L_1$ of index 4. The only way all these indices can be achieved is if $P_2 \in L_1 \setminus L_2$, D is tangent to L_2 at P_3 and the tangent line to D at P_1 is different from L_1 and L_2;

$((c), (a))$ In this case we have a geometric configuration giving a pencil generated by curves $Q_1 + S_1$ and $Q_2 + S_2$ where the intersection point P_1 of Q_1 and S_1 is a base point of index 6. The only way of obtaining this index is if Q_1 and Q_2 intersect at P_1 with index 4;

$((c),(b))$ In this case we have a geometric configuration giving a pencil generated by curves $Q+S$ and D where the intersection point P_1 of Q and S is a base point of index 6 and the node P_2 of D is a base point of index 2. The only way these indices can be achieved is if $P_2 \in Q \setminus S$ and D and Q intersect at P_1 with index 4;

$((d),(a))$ In this case we have curves $L_1+L_2+L_3$ and $Q+S$ where the common intersection point P_1 of L_1, L_2 and L_3 should be a base point of index 5 which can not be achieved on a smooth point of $Q+S$;

$((d),(b))$ In this case we have a geometric configuration giving a pencil generated by curves $L_1+L_2+L_3$ and D where the common intersection point P_1 of L_1, L_2 and L_3 is a base point of index 5 and the node P_2 of D is a base point of index 2. To obtain these indices we need P_1 to be the flex point of D and one of the three lines, say L_1, is its inflectional line and P_2 is a point on another line, say L_2;

Now we will show that it is possible to impose some restrictions to each configuration giving a pencil so that it will contain a cubic curve yielding the extra special fiber of type I_2. This will again be achieved applying Max Noether's Fundamental Theorem. We will relabel the generators and the base points according to the pencil they are in for further use.

The respective pencils and their restrictions are:

- $\Lambda_1 = \langle D, L_1+2L_2\rangle$, with D and L_1+2L_2 as in $(I_2,(b))$ and $(I_2^*,(b))$, respectively, where the line L_1 intersects the nodal cubic curve D at the node $P_1^{(1)}$ and another point $P_2^{(1)}$ and L_2 is a line through $P_2^{(1)}$ and tangent to D at a third point $P_3^{(1)}$. We need the following restriction: $P_2^{(1)}$ must be the inflectional point of D. In this way we shall have a member of Λ_1 as in $(I_2,(a))$ where the line S is the inflectional line of D at $P_2^{(1)}$ and the conic curve Q is the one tangent to L_1 at $P_1^{(1)}$ and intersecting D with index 4 at $P_3^{(1)}$. The configuration of generators and base points can be seen in Figure 4.22.

- $\Lambda_2 = \langle Q+S, L_1+2L_2\rangle$, with $Q+S$ and L_1+2L_2 as in $(I_2,(a))$ and $(I_2^*,(b))$, respectively, where the lines S, L_1 and L_2 meet at a point $P_1^{(2)}$ and the irreducible conic Q is tangent to L_2 at a point $P_2^{(2)}$. We need the following restriction: Q must be tangent to L_1 at some point $P_3^{(2)}$. In this way we shall have in Λ_2 a member D as in $(I_2,(b))$ where $P_3^{(2)}$ is the node, $P_1^{(2)}$ is the flex point and S is the inflectional line and D is tangent to L_2 at $P_2^{(2)}$.

- $\Lambda_3 = \langle Q'+S', R_1+2R_2\rangle$, with $Q'+S'$ and R_1+2R_2 as in $(I_2,(a))$ and $(I_2^*,(a))$, respectively, where the irreducible conic curve Q' is tangent to the line R_1 at the intersection point $P_1^{(3)}$ of R_1 and R_2, Q' intersects R_2 at another point $P_2^{(3)}$, while the line S' intersects R_2 at a third point $P_3^{(3)}$ and intersects R_1 at $P_4^{(3)}$. We need the following restriction: S' must be chosen in a way that the tangent line to Q' at $P_2^{(3)}$ also intersects R_1 at $P_4^{(3)}$. In this way have another member $Q''+S''$ of Λ_3 as in $(I_2,(a))$ where the line S'' is the tangent line to Q' at $P_4^{(3)}$ and the conic curve Q'' is the one that intersects Q' at $P_1^{(3)}$ with index 4 and is tangent to S' at $P_3^{(3)}$. The configuration of generators and base points can be seen in Figure 4.23.

- $\Lambda_4 = \langle D', Q_1+S_1\rangle$, with D' and Q_1+S_1 as in $(I_2,(b))$ and $(I_2^*,(c))$, respectively, where the nodal cubic curve D' intersects the irreducible conic Q_1 with index 4 at $P_1^{(4)}$, the tangency point of Q_1 and S_1; the node $P_2^{(4)}$ of D' is in Q_1; S_1 intersects D' at another point $P_3^{(4)}$. We need the following restrictions: the point $P_1^{(4)}$ in D' must be chosen in a way that both tangent lines to Q_1 at $P_1^{(4)}$ and $P_2^{(4)}$ meet D' at $P_3^{(4)}$. Thus we must have a member of Λ_4 as in $(I_2,(a))$ where the conic curve Q''' is the one intersecting D' and Q_1 at $P_1^{(4)}$ with indices 6 and 4, respectively, and the line S''' is the tangent line to Q_1 at $P_2^{(4)}$. The configuration of generators and base points can be seen in Figure 4.24.

- $\Lambda_5 = \langle Q''' + S''', Q_1 + S_1 \rangle$, with $Q''' + S'''$ and $Q_1 + S_1$ as in $(I_2,(a))$ and $(I_2^*,(c))$, respectively, where the irreducible conic curve Q''' intersects Q_1 with index 4 at $P_1^{(5)}$, the tangency point of Q_1 and S_1; the line S''' intersects S_1 at another point $P_2^{(5)}$. We need the following restriction: S''' must be tangent to Q_1 at a point $P_3^{(5)}$. Thus we shall have a member of Λ_5 as in $(I_2,(a))$ where the node is the point $P_3^{(5)}$.
- $\Lambda_6 = \langle D'', L_3 + L_4 + L_5 \rangle$ with D'' and $L_3 + L_4 + L_5$ as in $(I_2,(b))$ and $(I_2^*,(d))$, respectively, where the nodal cubic curve D'' has node $P_1^{(6)}$ over L_3 and the intersection point $P_2^{(6)}$ of the lines L_3, L_4, L_5 is the flex point of D'' and L_4 is its inflectional line. We need the following restriction: L_5 must be the line through $P_2^{(6)}$ that is tangent to D'' at another point $P_3^{(6)}$. In this way Λ_6 has another member D''' as in $(I_2,(b))$ with node $P_3^{(6)}$. The configuration of generators and base points can be seen in Figure 4.25.

Notice that a pencil with generators as in Λ_2 can be turned into a pencil of the form Λ_1 via a Möbius transformation. We have the same relation between Λ_4 and Λ_5. Thus we only need to show the contractions for the pencils Λ_3, Λ_4 and Λ_6.

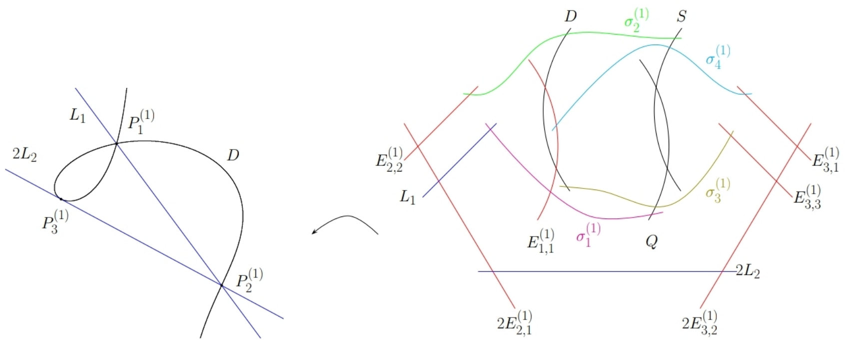


Figure 4.22: Resolution of Λ_1

In the resolution of Λ_1 the section $\sigma_4^{(1)}$ is the proper transform of the line through $P_1^{(1)}$ and $P_3^{(1)}$. In the resolution of Λ_4 the section $\sigma_4^{(4)}$ is the proper transform of the line through $P_1^{(4)}$ and $P_2^{(4)}$. In the resolution of Λ_6 the section $\sigma_4^{(6)}$ is the proper transform of the line through $P_1^{(6)}$ and $P_3^{(6)}$.

Now we can follow the sequences below to find pencils with the same configuration of Λ_1.

For Λ_3 we have the sequence

$$\sigma_1^{(3)} \to Q' \to \sigma_4^{(3)} \to R_1 \to E_{1,2}^{(3)} \to \sigma_3^{(3)} \to E_{3,1}^{(3)} \to R_2 \to E_{2,1}^{(3)}.$$

For Λ_4 we have the sequence

$$\sigma_4^{(4)} \to E_{2,1}^{(4)} \to \sigma_3^{(4)} \to S_1 \to E_{1,2}^{(4)} \to \sigma_1^{(4)} \to E_{1,5}^{(4)} \to E_{1,4}^{(4)} \to Q_1.$$

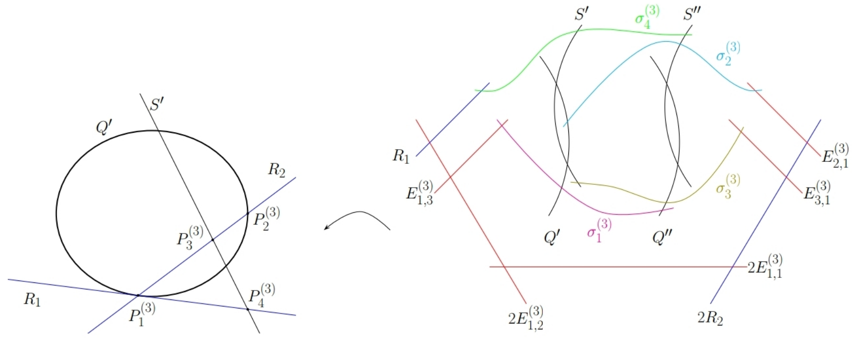


Figure 4.23: Resolution of Λ_3

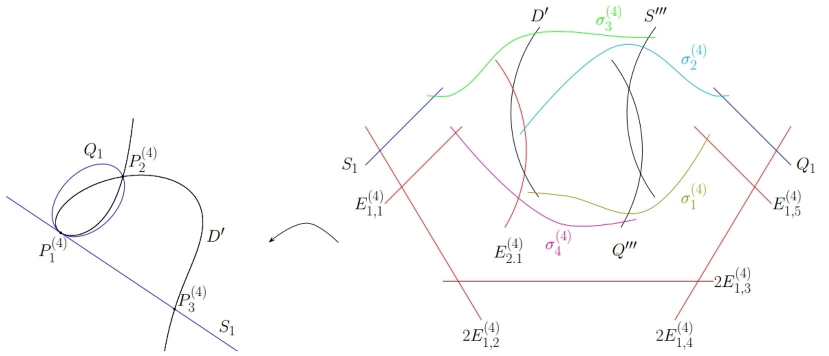


Figure 4.24: Resolution of Λ_4

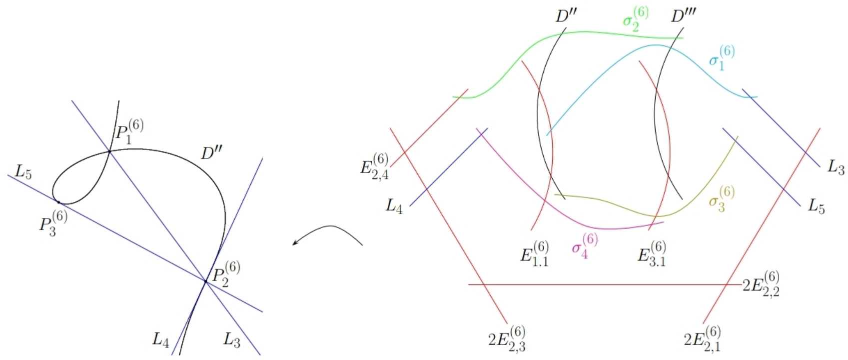


Figure 4.25: Resolution of Λ_6

For Λ_6 we have the sequence

$$\sigma_4^{(6)} \to E_{1,1}^{(6)} \to \sigma_2^{(6)} \to E_{2,4}^{(6)} \to E_{2,3}^{(6)} \to \sigma_3^{(6)} \to L_5 \to E_{2,1}^{(6)} \to L_3.$$

After these contractions we obtain the pencils $\langle S', E_{1,3}^{(3)} + 2E_{1,1}^{(3)} \rangle$, $\langle D', E_{1,1}^{(4)} + 2E_{1,3}^{(4)} \rangle$ and $\langle D'', L_4 + 2E_{2,2}^{(6)} \rangle$, respectively, all having the same geometric configuration of Λ_1. The irreducible nodal cubic curves are, respectively, the images of S', D' and D''. $\square$

Theorem 4.7. *The sets of equivalence classes $\mathscr{E}_{0,\ell}$ are singletons consisting of the equivalence class of the resolution of base points of the following pencils.*

($\mathscr{E}_{0,1}$) $\langle z^2x - y^3 - y^2x, -x^3\rangle$;

($\mathscr{E}_{0,2}$) $\langle z^2x - y^3 - y^2x, -x(x+y)^2\rangle$;

($\mathscr{E}_{0,3}$) $\langle z^2x - y^3 - y^2x, -x^2y\rangle$;

($\mathscr{E}_{0,4}$) $\langle z^2x - y^3 - y^2x, -x(xz + yz - xy)\rangle$;

($\mathscr{E}_{0,5}$) $\langle z^2x - y^3 - y^2x, -xz(x+y)\rangle$;

($\mathscr{E}_{0,6}$) $\langle z^2x - y^3 - y^2x, -y(x+y)^2\rangle$;

($\mathscr{E}_{0,7}$) $\langle z^2x - y^3 - y^2x, -x(y^2 - xy - xz)\rangle$;

($\mathscr{E}_{0,8}$) $\langle (xz + x^2 - y^2)z, -(xz - x^2 + y^2)y\rangle$.

Our proof is a case by case analysis and we explain the general strategy, by splitting it in two situations. The first is in the presence of an irreducible nodal cubic curve as the first generator of the pencil, which occur in the first seven pencils in Theorem 4.6. In this event, we show that up to an automorphism of $\mathbb{P}^2$, the second generator is uniquely determined by the first generator.

The second situation occurs only for the last pencil, where such a nodal cubic curve does not appear. In this case we will apply Nagell's algorithm ([**C**] and references therein) to show that a Weierstrass form for each pencil as in item 8 of Theorem 4.6 is isomorphic to a Weierstrass form for the specific pencil stated above. Since we need to apply Nagell's algorithm we decided to summarize it in the following remark.

Remark 4.8. *Without loss of generality, we may assume that* $P = (0 : 0 : 1)$ *is a point of the general member* $H = F - TG$ *of the pencil* $\langle F, G\rangle$, *which is an irreducible curve over* $K = k(T)$. *Hence we can write* $H = H_3 + H_2z + H_1z^2$ *with* $H_1 = s_8x + s_9y$, $H_2 = s_5x^2 + s_6xy + s_7y^2$ *and* $H_3 = s_1x^3 + s_2x^2y + s_3xy^2 + s_4y^3$, *where* H_1 *is the tangent of* H *at* P *and* s_8 *is assumed to be different from* 0. *We consider the second common point* $Q = (-e_2s_9 : e_2s_8 : e_3)$ *between* H_1 *and* H, *where* $e_i = H_i(s_9, -s_8)$ *for* $i = 2, 3$. *If necessary we can change* $z \mapsto y + z$, *which fixes* P *and* H_1, *to assume* $e_3 \neq 0$. *We make the change of variables*

$$x = x_1 - s_9(e_2/e_3)z_1,\ \ y = y_1 + s_8(e_2/e_3)z_1,\ \ z = z_1$$

which sends $(0 : 0 : 1)$ *to* Q *and* $(e_2s_9 : -e_2s_8 : e_3)$ *to* P. *Then, by doing*

$$x_1 = x_2 + s_9y_2,\ \ y_1 = -s_8y_2,\ \ z_1 = z_2$$

we change the tangent at Q *to* x_2. *In this system of coordinates* H *is sent to* $H' = H'_3 + H'_2z_2 + H'_1z_2^2$, *where* $H'_i \in K[x_2, y_2]$ *is homogeneous of degree* i. *Writing* $h = H'(x_2, y_2, 1)$, $h_i = H'_i(x_2, y_2, 1)$ *and using the blow-up*

$$x_2 = x_2,\ \ y_2 = ux_2$$

at $Q = (0, 0)$ *we obtain the curve* $x_2^2h_3(1, u) + x_2h_2(1, u) + h_1(1, u)$, *birationally equivalent to* H. *If we define*

$$v = -h_3(1, u)x_2 + h_2(1, u)$$

we get a Weierstrass form given by $v^2 - (h_2(1, u)^2 - h_1(1, u)h_3(1, u))$, *which is also birationally equivalent to* H.

Proof. As we indicated in the strategy of our proof, in the seven first pencils of Theorem 4.6, the irreducible nodal cubic curve D can be taken as $z^2x - y^3 - y^2x$, up to change of coordinates of $\mathbb{P}^2$.

$\boxed{\mathscr{E}_{0,1}}$ In this case the inflectional line L is given by x. So $\mathscr{E}_{0,1}$ is the class given by the resolution of the pencil $\langle z^2x - y^3 - y^2x, -x^3\rangle$.

$\boxed{\mathscr{E}_{0,2}}$ In this case the inflectional line L_1 is given by x and the line L_2 is given by $x + y$ since this is the tangent line at the unique (non-flex) smooth point $(1 : -1 : 0)$ of the intersection of D and its polar curve at the flex $(0 : 0 : 1)$. So $\mathscr{E}_{0,2}$ is the class given by the resolution of the pencil $\langle z^2x - y^3 - y^2x, -x(x + y)^2\rangle$.

$\boxed{\mathscr{E}_{0,3}}$ In this case, the line L_1 trough the node $(1 : 0 : 0)$ and the flex is given by y and the inflectional line L_2 is given by x. So $\mathscr{E}_{0,3}$ is the class given by the resolution of the pencil $\langle z^2x - y^3 - y^2x, -x^2y\rangle$.

$\boxed{\mathscr{E}_{0,4}}$ In this case the inflectional line S_1 is given by x and the irreducible conic Q_1 with the required intersection indices can be either $xz + yz - xy$ or $xz + yz + xy$. Since both conic curves can be obtained from each other via the $\mathbb{P}^2$-automorphism $z \mapsto -z$, which fixes the curve D, we can choose Q_1 to be given by the first one. So $\mathscr{E}_{0,4}$ is the class given by the resolution of the pencil $\langle z^2x-y^3-y^2x, -x(xz+yz-xy)\rangle$.

$\boxed{\mathscr{E}_{0,5}}$ In this case L_1 is given by x and L_2 is given by $x + y$ and L_3 is given by z. So $\mathscr{E}_{0,5}$ is the class given by the resolution of the pencil $\langle z^2x - y^3 - y^2x, -xz(x + y)\rangle$.

$\boxed{\mathscr{E}_{0,6}}$ In this case L_1 is given by y and L_2 is given by $x+y$. So $\mathscr{E}_{0,6}$ is the class given by the resolution of the pencil $\langle z^2x - y^3 - y^2x, -y(x + y)^2\rangle$.

$\boxed{\mathscr{E}_{0,7}}$ In this case S_1 is given by x and Q_1 can be either $y^2-xy-xz$ or $y^2-xy+xz$. Since both conic curves can be obtained from each other via the $\mathbb{P}^2$-automorphism $z \mapsto -z$, we can choose Q_1 to be given by the first one. So $\mathscr{E}_{0,7}$ is the class given by the resolution of the pencil $\langle z^2x-y^3-y^2x, -x(y^2-xy-xz)\rangle$.

Now we will study the only case that does not have a nodal cubic curve in the pencil codifying the prescribed geometric properties of the required fibration.

$\boxed{\mathscr{E}_{0,8}}$

Up to a projective change of coordinates, L_1, L_2 and the tangent line to Q_1 at $P_1^{(1)}$ are given by y, z and x, respectively. With restrictions as in item 8 of Theorem 4.6, Q_1 and Q_2 are given respectively by $-x^2 + a^2y^2 + bxz$ and $x^2 - a^2y^2 + bxz$, with $ab \neq 0$. Therefore each fibration with $\mathcal{T} = (A_3 \oplus A_1)^{\oplus 2}$ is equivalent to a fibration obtained from the pencil $\langle z(x^2 - a^2y^2 + bxz), -y(-x^2 + a^2y^2 + bxz)\rangle$. If we apply Nagell's algorithm 4.8 to the curve $H_{a,b} = z(x^2 - a^2y^2 + bxz) + T(y(-x^2 + a^2y^2 + bxz))$ we obtain a birationally equivalent curve given by the Weierstrass form

$$v^2 - (a^2b^4T^3u^3 + (b^4T^4 + a^2b^2T^2)u^2 + b^2T^3u)$$

having j-invariant

$$(b^{12}T^{12} - a^6b^6T^6 + a^{12})/(a^4b^8T^8 + a^6b^6T^6 + a^8b^4T^4).$$

If we also apply Nagell's algorithm to the curve $H_{1,1}$ followed by the Möbius transformation sending T to $\dfrac{b}{a}T$ we will obtain a Weierstrass form with $(a^2b^4T^4 + a^4b^2T^2)$ being the coefficient of u^2 and with the same j-invariant as above. By the Remark 3.11 the elliptic curves given by these Weierstrass forms are isomorphic. So $\mathscr{E}_{0,8}$ is the class given by the resolution of the pencil $\langle z(x^2 - y^2 + xz), -y(-x^2 + y^2 + xz)\rangle$. □

Example 4.9. *We consider the pencils* $\Lambda_1 = \langle -x(y + x)(z - y), (x - y)(x^2 - xy - xz + z^2)\rangle$ *and* $\Lambda_2 = \langle -x(\alpha y + x)(z - y), (x - y)(x^2 - xy - xz + z^2)\rangle$ *where* α *is a primitive element for the field* *extension* $\mathbb{F}_{3^8}|\mathbb{F}_3$. *Both pencils give rise to elliptic surfaces having the special fibers* $2I_4, I_2, 2I_1$. *This means that their discriminants,* Δ_1 *and* Δ_2*, must have two roots of order* 4*, one of order* 2 *and two of order* 1. *If they were equivalent there should exist a Möbius transformation sending* Δ_1 *to* Δ_2. *Since such a transformation preserves the orders of roots, it should be determined by its action on roots of order* 2 *and* 4*, resulting in two possibilities. However, a* MAGMA *check shows that neither sends the roots of order* 1 *of* Δ_1 *to those of* Δ_2. *Our computations suggest that the space of non-equivalent elliptic fibrations having this configuration of special fibers have positive dimension.*

CHAPTER **5**

Vector fields on the projective plane inducing fibrations by singular curves

Let us consider k an algebraically closed field of characteristic three. In this chapter we will deal with the question:

Question 5.1. *Given a pencil of generically smooth plane cubic curves $(F : G)\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ how can we build a 3-closed vector field D on $\mathbb{P}^2$ such that the rational map $(\mathbb{P}^2)^D \dashrightarrow \mathbb{P}^1$ induces a fibration by singular curves of genus two?*

In this chapter we will answer Question 5.1 for the pencils described in Theorem 4.7, which is enough to answer for all pencils inducing fibrations by elliptic curves with Mordell-Weil rank zero.

Given a fibration $f\colon S \to \mathbb{P}^1$ by singular curves of genus two whose smoothing is a fibration by elliptic curves $f_1\colon S_1 \to \mathbb{P}^1$, we will construct a 3-closed vector field D_1 on S_1 such that we have a commutative diagram

$$\begin{tikzcd} S = S_1^{D_1} \arrow[d, "f"] & S_1 \arrow[l, "\pi^{D_1}"'] \arrow[d, "f_1"] \\ \mathbb{P}^1 & \mathbb{P}^1 \arrow[l, "F_{\mathbb{P}^1,k}"'] \end{tikzcd}$$

where $F_{\mathbb{P}^1,k}\colon \mathbb{P}^1 \to \mathbb{P}^1$ is the relative Frobenius map of $\mathbb{P}^1$. More precisely we will find a k-derivation D_1 of $k(S_1)$ whose kernel is $k(S_1)^{D_1} = k(S)$. Just to refresh our notations, the field of rational functions of the source and the target of $F_{\mathbb{P}^1,k}$ are $k(T)$ and $k(t)$, respectively, where T is a transcendental element over k and $t = T^3$.

It follows from Proposition 3.5 that $k(S) = k(x, y, t)$ and $k(S_1) = k(z, w, T)$ where $y^2 = h(x^3 - j^3)(x^3 - j^3x - j^3)$, $w^2 = -h(j^4z^3 + j^3z^2 - 1)$, $h \in k(t)$ and $j \in k(T) \setminus k(t)$. It also follows from the proof of Proposition 3.5 that $z = \frac{1}{x-j}$ and $w = z^3y$.

Lemma 5.2. *With the above notations, we have $k(S) = k(z^3, w, T^3)$.*

Proof. Notice that $T^3 = t \in k(S)$. Moreover, $z^3, w \in k(S)$, since $z^3 = \frac{1}{x^3-j^3}$ and $j^3 \in k(S)$. Hence $k(S) \supseteq k(z^3, w, T^3)$.

For the other inclusion, since $t = T^3$ and $y = \frac{w}{z^3}$ we have $t, y \in k(z^3, w, T^3)$ and it remains to prove that $x \in k(z^3, w, T^3)$. From the equation relating x and y, namely $y^2 = h(x^3 - j^3)(x^3 - j^3x - j^3) = hz^{-3}(z^{-3} - j^3x)$, we obtain $x = z^{-3} - z^3y^2h^{-1}$ with $h \in k(T^3)$. $\square$

This lemma motivates us to look for a k-derivation D_1 of $k(z, w, T)$ whose kernel is $k(z^3, w, T^3)$. A natural candidate is to consider the extension to $k(z, w, T)$ of the derivation $\frac{\partial}{\partial z}$ of $k(z, w)$. To define such extension we just need to determine the derivation applied to T and we do this by considering the relation $w^2 = -h(j^4z^3 + j^3z^2 - 1)$. To this end, if we write $f(z, T) = h(j^4z^3 + j^3z^2 - 1)$, then D_1 is defined by the k-derivation satisfying $D_1(z) = 1$, $D_1(w) = 0$ and such that

$$0 = 2wD_1(w) = -(f_z + f_TD_1(T))$$

where f_z and f_T are the partial derivatives of f with respect to indicated variables in $k(z, T)$. Since $h \in k(T^3)$ it follows that $f_T = hj^3j'z^3$ where j' is the derivative of j with respect to T in $k(T)$. On the other hand $j \in k(T) \setminus k(T^3)$ implies that $j' \neq 0$, that is, $f_T \neq 0$. Hence $D_1(T)$ can be defined by $-\frac{f_z}{f_T}$ and, therefore

$$D_1(g(z, w, T)) = g_z - \frac{f_z}{f_T}g_T = \frac{g_zf_T - f_zg_T}{f_T},$$

for every $g(z, w, T) \in k(z, w, T)$.

Lemma 5.3. *If $D_1 \in \mathrm{Der}_k(k(z, w, T))$ is defined as above, then $\operatorname{Ker} D_1 = k(z^3, w, T^3)$.*

Proof. The definition of D_1 implies $k(z^3, w, T^3) \subseteq \operatorname{Ker} D_1 \subseteq k(z, w, T)$. Moreover, from the previous lemma $k(z, w, T)|k(z^3, w, T^3)$ is a field extension of degree 3 because $k(z, w, T) = k(x, y, t)k(T)$. Hence we may conclude $k(z^3, w, T^3) = \operatorname{Ker} D_1$ just observing that $D_1(z) \neq 0$. $\square$

Lemma 5.4. *The derivation D_1 satisfies $D_1^3 = 0$ and, in particular, it is 3-closed.*

Proof. Since D_1^3 is also a k-derivation of $k(z, w, T)$ we have $D_1^3(g(z, w, T)) = g_zD_1^3(z) + g_wD_1^3(w) + g_TD_1^3(T)$ for each $g(z, w, T) \in k(z, w, T)$. Hence, it is enough to prove that D_1^3 kills z, w and T. Clearly $D_1^3(z) = D_1^3(w) = 0$. By direct computation we also check $D_1^3(T) = 0$. Indeed

$$D_1(T) = \frac{hj^3z}{hj^3j'z^3} = \frac{1}{j'z^2} \text{ and } D_1^2(T) = D_1\left(\frac{1}{j'z^2}\right) = -\frac{j''z^2D_1(T) + j'^2z}{(j'z^2)^2} = \frac{-j''z^2 + (j')^2z^3}{(j'z^2)^3}$$

where j'' is the second derivative of j with respect to T in $k(T)$. Therefore,

$$D_1^3(T) = D_1\left(\frac{-j''z^2 + (j')^2z^3}{(j'z^2)^2}\right) = \frac{-j'''D_1(T)z^2 + j''z - j'j''D_1(T)z^3}{(j'z^2)^3} = \frac{j''z - j''z}{(j'z^2)^3} = 0$$

since the third derivative j''' of j is zero. $\square$

We summarize all facts above in the following result.

Proposition 5.5. *Let $f_1 \colon S_1 \to \mathbb{P}^1$ be the smoothing of an absolutely elliptic fibration $f \colon S \to \mathbb{P}^1$ by genus two singular curves, over an algebraically closed field k of characteristic three. Let $k(S_1) = k(z, w, T)$, with z, w, T satisfying the Weierstrass polynomial of Proposition 3.5 for $K = k(t)$ and $L = k(T)$. Then the 3-closed vector field D_1 such that $S = S_1^{D_1}$ is given by*

$$D_1(g) = \frac{f_Tg_z - f_zg_T}{f_T},$$

for every $g \in k(z, w, T)$, where $f(z, T) = h(j^4z^3 + j^3z^2 - 1)$ and f_z, f_T, g_z, g_T stand by the partial derivatives of f and g with respect to the indicated variables.

Now we will use the derivation D_1 in order to answer Question 5.1 for each pencil appearing in Theorem 4.7.

Theorem 5.6. *Let Λ_ℓ be the pencil in item ℓ of Theorem 4.7, for each $\ell = 1, \ldots, 8$. If $D_{1,\ell} = A_\ell \partial/\partial x + B_\ell \partial/\partial y$ is a 3-closed vector field, with coordinates x, y in the chart $z \neq 0$ in $\mathbb{P}^2$, whose induced fibration on $(\mathbb{P}^2)^{D_{1,\ell}}$ is by genus two singular curves, then A_ℓ and B_ℓ can be chosen as below.*

1. $A_1 = x^2$ *and* $B_1 = y^2$;
2. $A_2 = x^6 + x^4y + x^3y^2 - x^3y + x^3 + x^2y^2 - x^2 + xy^3 - xy^2 - y^3 + y^2$

   and

   $B_2 = (y-1)^2(x^3 - xy + x - y^2 + y)$;
3. $A_3 = -x$ *and* $B_3 = (x+y)(x-y^3)$;
4. $A_4 = -(x^4y^2 + x^4y + x^4 + x^3y^4 + x^3y^3 + x^3y^2 + x^3y - x^3 - x^2y^5 + x^2y^4 + x^2 + xy^6 - xy^5 - y^6)$

   and

   $B_4 = -(x^2y^5 - x^2y^4 + x^2y^3 + x^2y^2 + x^2 - xy^6 - xy^5 - xy^3 + xy^2 - xy + y^7 - y^6 - y^5)$;
5. $A_5 = -(x^4 + x^3y^3 + x^3y + x^2y^4 + x^2 - xy^5 - y^6)$

   and

   $B_5 = x^2y^4 + x^2y^2 - x^2 - xy^5 + xy^3 + xy + y^6$;
6. $A_6 = -x^2$ *and* $B_6 = y^2$;
7. $A_7 = -(x^2y^2 + x^2y + xy^3 - x + y^3)$

   and

   $B_7 = -(y+1)(y-1)(xy + x + y^2 - y)$;
8. $A_8 = -xy(x+y)(x-y)$ *and* $B_8 = (x^2 - x^2y^2 + y^4)$.

Proof. Our strategy here will be similar to the one used in case $\ell = 8$ of Theorem 4.7. In this way we will take, in each item of the mentioned theorem, a general member $F - TG$ of the pencil and we apply Nagell's algorithm (see Remark 4.8) to reach a Weierstrass equation $v^2 = a_0u^3 + a_2u^2 + a_4u + a_6$ for the generic fiber of the fibration induced by the pencil. After that we apply the map

$$z_1 = a + bu, \quad w_1 = ev$$

where $a = a_4/a_2, b = ja_2/a_0$ and $e = a_0^2 j$, as in Lemma 3.13, to obtain another Weierstrass form for the generic fiber as in Proposition 3.5 - with z_1 and w_1 instead of z and w - since these are the coordinates where D_1 is known. To see the derivation D_1 of $k(z_1, w_1, T)$ as a derivation of $k(u, v, T)$ we only need to express $D_1(u), D_1(v)$ and $D_1(T)$ in terms of the generators u, v and T of the function field $k(S_1)|k$. We proceed similarly in each step of Nagell's algorithm until reaching the derivation D_1 expressed in the coordinates x, y and T or equivalently only in x, y since $T = F(x,y,1)/G(x,y,1)$. Along the previous process, whenever we find it convenient, we substitute the derivation by a rational multiple in order to get simpler expressions in each step. All computations were performed using the algebra systems Magma [**BCP**] and Maxima [**Max**].

Since each case has computations following the same pattern, we decided to show them only in case 8.

Applying Nagell's algorithm to the pencil $\langle (xz+x^2-y^2)z, -(xz-x^2+y^2)y\rangle$, obtained in Theorem 4.7, we get a Weierstrass form with $a_0 = T^3, a_2 = T^4+T^2, a_4 = T^3, a_6 = 0$ and j-invariant $(T^{12}-T^6+1)/(T^8+T^6+T^4)$. After the map indicated at the beginning of the proof we get a normal form over the field $k(z_1, w_1, T)$ where

$$\begin{gathered} D_{1,8}(z_1) = 1, \\ D_{1,8}(w_1) = 0, \\ D_{1,8}(T) = 1/(j' z_1^2) = (T^{13}-T^{11}-T^7+T^5)/((T^{16}-T^{12}-T^{10}+T^6+T^4-1)z_1^2). \end{gathered}$$

Then we take the equivalent derivation, also denoted $D_{1,8}$, given by

$$\begin{gathered} D_{1,8}(z_1) = (T^{16}-T^{12}-T^{10}+T^6+T^4-1)z_1^2, \\ D_{1,8}(w_1) = 0, \\ D_{1,8}(T) = T^{13}-T^{11}-T^7+T^5. \end{gathered}$$

Taking the inverse map we can compute $D_{1,8}(u), D_{1,8}(v)$ and $D_{1,8}(T)$ in terms of u, v, T. They have $T^4(T^2-1)^3$ as a common factor and cleaning up this factor we obtain an equivalent derivation given by

$$\begin{gathered} D_{1,8}(u) = Tu^2-(T^2+1)u, \\ D_{1,8}(v) = (T^2+1)v, \\ D_{1,8}(T) = T^3-T. \end{gathered}$$

Writing $D_{1,8}$ in the next generators of $k(S_1)$ we see that $T(u^2-1)$ is the denominator of $D_{1,8}(x_2)$. Multiplying by this polynomial we obtain an equivalent derivation given by

$$\begin{gathered} D_{1,8}(u) = T^2u^4+(-T^3-T)u^3-T^2u^2+(T^3+T)u, \\ D_{1,8}(x_2) = (-T^3u^2+T^2u-T)x_2-Tu^2+(-T^2-1)u-T, \\ D_{1,8}(T) = (T^4-T^2)u^2-T^4+T^2. \end{gathered}$$

In terms of the next generators x_2, y_2 and T, x_2^3 is the common denominator of $D_{1,8}$. Hence the equivalent derivation is given by

$$\begin{gathered} D_{1,8}(x_2) = (-T^3x_2^2-Tx_2)y_2^2+(T^2x_2^3+(-T^2-1)x_2^2)y_2-Tx_2^4-Tx_2^3, \\ D_{1,8}(y_2) = T^2y_2^4+((T^3-T)x_2-T)y_2^3+(-T^2-1)x_2y_2^2+(T^3x_2^3-Tx_2^2)y_2, \\ D_{1,8}(T) = (T^4-T^2)x_2y_2^2+(T^2-T^4)x_2^3. \end{gathered}$$

For the generators x_1, y_1, T we have

$$\begin{gathered} D_{1,8}(x_1) = (-T^3x_1^2-Tx_1)y_1^2+((T^2+1)x_1^2-T^2x_1^3)y_1-Tx_1^4-Tx_1^3, \\ D_{1,8}(y_1) = -T^2y_1^4+((T^3-T)x_1-T)y_1^3+(T^2+1)x_1y_1^2+(T^3x_1^3-Tx_1^2)y_1 \\ D_{1,8}(T) = (T^4-T^2)x_1y_1^2+(T^2-T^4)x_1^3. \end{gathered}$$

Finally we have to determine $D_{1,8}$ in terms of x, y, T or equivalently in terms of x, y, because $T = F(x,y,1)/G(x,y,1)$. With these coordinates we get $x^3y(x^2-y^2)(x+y+1)^2(x-y+1)^2$ as a common factor of $D_{1,8}(x)$ and $D_{1,8}(y)$. Therefore $D_{1,8}$ is equivalent to a derivation given by

$$A_8 = D_{1,8}(x) = -xy(x+y)(x-y) \text{ and } B_8 = D_{1,8}(y) = (x^2-x^2y^2+y^4).$$

□

APPENDIX **A**

Blow-ups of smooth surfaces

In this appendix we will remember some facts about blow-ups of a smooth surface S at point P and its relation with a pencil of curves in S (cf. [**H**] and [**Mi**]).

Let S be a smooth surface and let $P \in S$ be a closed point. Then there exist a surfaces $\tilde{S}$ and a morphism $\pi\colon \tilde{S} \to S$, which are unique up to isomorphism, such that

1. $\pi \mid_{\pi^{-1}(S-\{P\})} \colon \pi^{-1}(S-\{P\}) \to S-\{P\}$ is an isomorphism;
2. $\pi^{-1}(P) = E$ is isomorphic to $\mathbb{P}^1$.

We say that π is the blow-up of S at P and we call E its exceptional curve.

Lemma A.1. *Let C be an irreducible curve on S that passes through P with multiplicity m. Then $\pi^*C = \tilde{C} + mE$, where $\tilde{C}$ is the closure of $\pi^{-1}(C-\{P\})$ in $\tilde{S}$ called the proper transform of C.*

Proposition A.2. *Let S be a smooth surface, $\pi\colon \tilde{S} \to S$ the blow-up of a point $P \in S$ and $E \subset \tilde{S}$ the exceptional curve. Let D, D' be divisors on S, then*

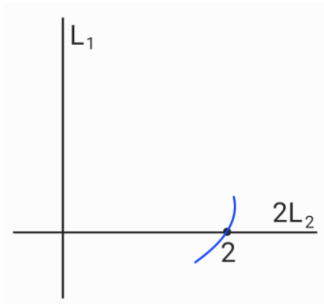
- $(\pi^*D) \cdot (\pi^*D') = D \cdot D'$;
- $E \cdot (\pi^*D) = 0$;
- $E^2 = -1$.

Proposition A.3. *Let $\{C_\lambda\}$ be a pencil curves (general member smooth) on a smooth surface S with a base point P. Let $\pi\colon \tilde{S} \to S$ be the blow-up of S at P, then*

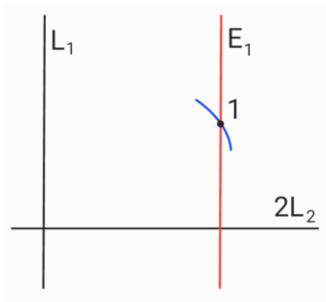
- *$\{\pi^*C_\lambda - E\}$ is the corresponding pencil on $\tilde{S}$;*
- *Given a member C_{λ_0} in S, the corresponding member in $\tilde{S}$ is $\tilde{C}_{\lambda_0} + (m_P(C_{\lambda_0}) - 1)E$;*
- *If P is a base point of $\{C_\lambda\}$ of index $n \geq 2$ then there is a point $P_1 \in E$ which is a base point of $\{\pi^*C_\lambda - E\}$ of index $n-1$;*
- *Let B be an irreducible component of C_{λ_0}, then $\tilde{B}^2 = B^2 - m_P(B)^2$;*
- *If P is a base point of index 1, then E is a global section in $\tilde{S}$, that is E is not a component of any curve in $\{\pi^*C_\lambda - E\}$ and intersects every curve of this pencil transversally.*

We also can use this result to conclude that in a rational elliptic surface every component of a reducible fiber has self-intersection equal to -2. Furthermore, if we look at these results from the view point of contractions of curves with self-intersection equal to -1, we see that the self-intersection of a curve of a pencil increase by as many as the number points this curve intersects the exceptional curve we contract. We also increase the base point index by one.

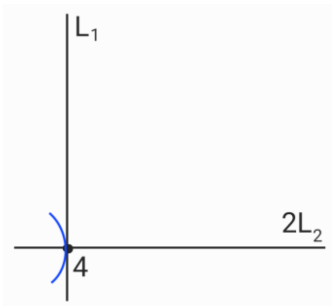
To illustrate how to obtain the special fibers from the singular curves in the Table 4.3, we will present the blow-ups of the curves in the row I_2^*. Since each marked singular point gives rise to a different sequence of blow-ups, which gives us disjoint sequences of new components (or sections when the index is equal to 1). We will see how the index and multiplicity of a point affect the new components. In each figure the blue curve represents locally a smooth member of a pencil containing the singular curve as a member. We will abuse notation and denote the proper transform of a curve C by C. We start with a point of index 2 of the singular $(I_2^*, (a))$.



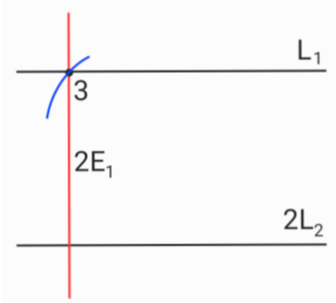
The base point is a point of index 2 and multiplicity 2 implies that the exceptional curve is a component of multiplicity 1 of the curve above this and we have a base point of index 1 in the exceptional curve, which give a section in the next blow-up.



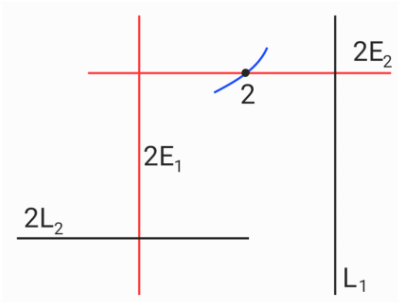
Now we look at the point of index 4 in $(I_2^*, (a))$.



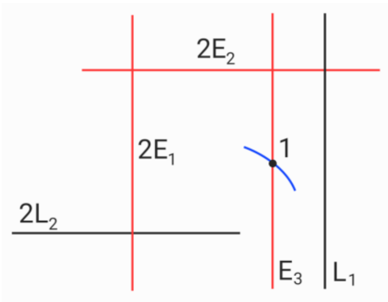
The base point is a point of index 4 and multiplicity 3 implies the exceptional curve is a component of multiplicity 2 of the curve above this and we have a base point of index 3 in the exceptional curve, since blue curve is tangent to L_1, this base point is also in the proper transform of L_1.



The base point is a point of index 3 and multiplicity 3 implies the exceptional curve is a component of multiplicity 2 of the curve above this and we have a base point of index 2 in the exceptional curve.

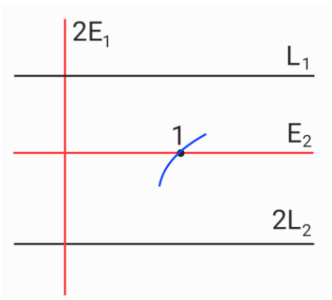


We have again a base point index 2 and multiplicity 2. So the next blow-up gives:

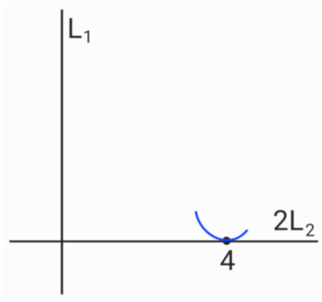


Putting all 3 sequences of blow-ups together we get essentially a fiber of type I_2^*, with possibly some base points that are smooth in this curve.

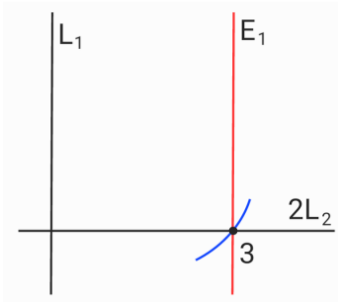
Next we have the singular curve $(I_2^*, (b))$. We already know that the base point of index 3 and multiplicity 3 will give the following configuration.



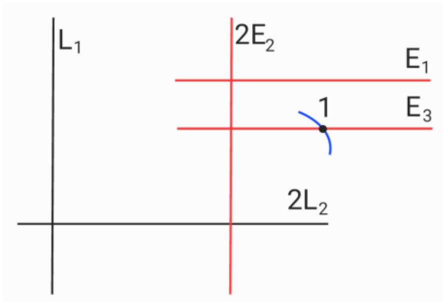
Now we look at the point of index 4 in $(I_2^*, (b))$.



The base point of index 4 and multiplicity 2 implies that the exceptional curve is a component of multiplicity 1 of the curve above this and we have a base point of index 3 in the exceptional curve, since blue curve is tangent to L_2, this base point is also in the proper transform of L_2.

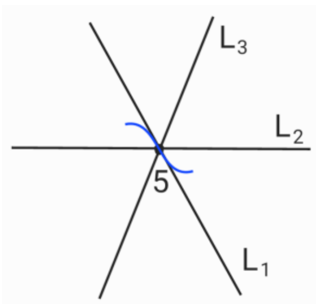


We have again a base point of index 3 and multiplicity 3. So the next blow-ups give:

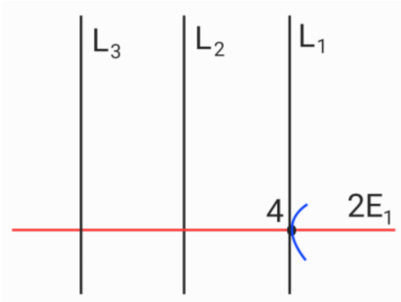


Putting these 2 sequences of blow-ups together we get essentially a fiber of type I_2^*.

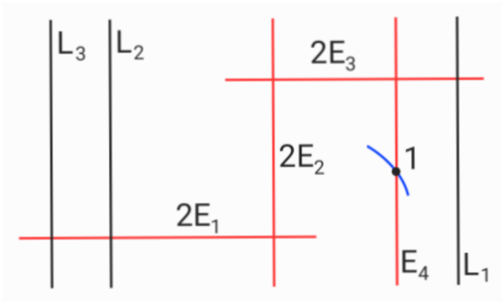
Next we have the singular curve $(I_2^*, (c))$.



The base point is a point of index 5 and multiplicity 3 implies the exceptional curve is a component of multiplicity 2 of the curve above this and we have a base point of index 4 in the exceptional curve, since blue curve has inflectional line L_1, its proper transform is tangent to the proper transform of L_2.

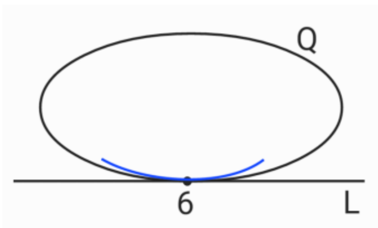


We already know that the base point of index 4 and multiplicity 3 will give the following configuration.

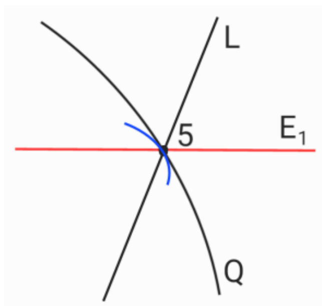


And this is the configuration of a fiber of type I_2^*.

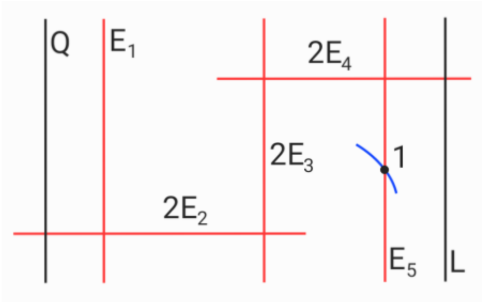
Finally we have the singular curve $(I_2^*, (d))$.



The base point is a point of index 6 and multiplicity 2 implies the exceptional curve is a component of multiplicity 1 of the curve above this and we have a base point of index 5 in the exceptional curve, since blue curve intersects the conic Q and the line with indices 4 and 2, respectively, its proper transform intersects the proper transforms of Q and L with indices 3 and 1, respectively.



And this is basically the configuration in the previous case, which gives our special fiber.



APPENDIX B

Codes

In this appendix we show the codes used on the algebra systems MAGMA and MAXIMA.

We start by showing the following code used on the algebra system MAXIMA to apply Naggel's algorithm in the case $\mathscr{E}_{0,8}$ which appears in the proof of Theorem 4.7.

(% i2)
```
F:y*(-x^2+a^2*y^2+b*x*z);
G:z*(x^2-a^2*y^2+b*x*z);
```

$$y\left(bxz+a^2y^2-x^2\right) \tag{\% o1}$$

$$z\left(bxz-a^2y^2+x^2\right) \tag{\% o2}$$

(% i3)
```
H:num(ratsimp(F+t*G));
```

$$btxz^2+\left(-a^2ty^2+bxy+tx^2\right)z+a^2y^3-x^2y \tag{\% o3}$$

(% i6)
```
H_3:coeff(H,z,0);
H_2:coeff(H,z,1);
H_1:coeff(H,z,2);
```

$$a^2y^3-x^2y \tag{\% o4}$$

$$-a^2ty^2+bxy+tx^2 \tag{\% o5}$$

$$btx \tag{\% o6}$$

(% i8)
```
s_8:coeff(H_1,x);
s_9:coeff(H_1,y);
```

$$b\,t \tag{\% o7}$$

$$0 \tag{\% o8}$$

(%i10) e_2:polymod(subst([x=s_9,y=-s_8],H_2),3);
e_3:polymod(subst([x=s_9,y=-s_8],H_3),3);

$$-a^2b^2t^3 \tag{\% o9}$$

$$-a^2b^3t^3 \tag{\% o10}$$

(%i14) x:x_1-s_9*(e_2/e_3)*z_1;
y:y_1+s_8*(e_2/e_3)*z_1;
z:z_1;
H_4:num(polymod(ratsimp(ev(H)),3));

$$x_1 \tag{\% o11}$$

$$tz_1 + y_1 \tag{\% o12}$$

$$z_1 \tag{\% o13}$$

$$\left(a^2t^2y_1 - btx_1\right) z_1{}^2 + \left(bx_1y_1 - a^2ty_1{}^2\right) z_1 + a^2y_1{}^3 - x_1{}^2y_1 \tag{\% o14}$$

(%i18) x_1:x_2+s_9*y_2;
y_1:-s_8*y_2;
z_1:z_2;
H_5:polymod(ratsimp(ev(H_4)),3);

$$x_2 \tag{\% o15}$$

$$-bty_2 \tag{\% o16}$$

$$z_2 \tag{\% o17}$$

$$\left(-a^2bt^3y_2 - btx_2\right) z_2{}^2 + \left(-a^2b^2t^3y_2{}^2 - b^2tx_2y_2\right) z_2 - a^2b^3t^3y_2{}^3 + btx_2{}^2y_2 \tag{\% o18}$$

(%i22) h:subst(z_2=1,H_5);
h_1:coeff(H_5,z_2,2);
h_2:coeff(H_5,z_2,1);
h_3:coeff(H_5,z_2,0);

$$-a^2b^3t^3y_2{}^3 - a^2b^2t^3y_2{}^2 + btx_2{}^2y_2 - b^2tx_2y_2 - a^2bt^3y_2 - btx_2 \tag{\% o19}$$

$$-a^2bt^3y_2 - btx_2 \tag{\% o20}$$

$$-a^2b^2t^3{y_2}^2 - b^2tx_2y_2 \tag{\% o21}$$

$$bt{x_2}^2y_2 - a^2b^3t^3{y_2}^3 \tag{\% o22}$$

(% i23) f:ratsimp(subst(y_2=u*x_2,h)/x_2);

$$\left(btu - a^2b^3t^3u^3\right){x_2}^2 + \left(-a^2b^2t^3u^2 - b^2tu\right)x_2 - a^2bt^3u - bt \tag{\% o23}$$

(% i24) v:-coeff(f,x_2,2)*x_2+coeff(f,x_2,1);

$$\left(a^2b^3t^3u^3 - btu\right)x_2 - a^2b^2t^3u^2 - b^2tu \tag{\% o24}$$

(% i25) W:polymod(ratsimp((coeff(f,x_2,1)^ 2-coeff(f,x_2,2)*coeff(f,x_2,0))),3);

$$a^2b^4t^4u^3 + \left(a^2b^2t^4 + b^4t^2\right)u^2 + b^2t^2u \tag{\% o25}$$

(% i26) Δ:-polymod(ratsimp(coeff(W,u,3)*coeff(W,u,1)^ 3
-coeff(W,u,1)^ 2*coeff(W,u,2)^ 2+coeff(W,u,0)*coeff(W,u,2)^ 3),3);

$$a^4b^8t^{12} + a^2b^{10}t^{10} + b^{12}t^8 \tag{\% o26}$$

(% i27) j:polymod(ratsimp(coeff(W,u,2)^ 6/(coeff(W,u,3)^ 2*Δ)),3);

$$\frac{a^{12}t^{12} - a^6b^6t^6 + b^{12}}{a^8b^4t^8 + a^6b^6t^6 + a^4b^8t^4} \tag{\% o27}$$

In the proof of Theorem 5.6 we analyze if we can swap a derivation with one of its rational multiples in order to get simpler expressions for D_1 applied to each system of generators of the function field. For instance, if $K(S_1) = K(x_1, x_2, x_3)$ and $D_1(x_1), D_1(x_2), D_1(x_3)$ are polynomials, we use the following code on the algebra systems MAGMA to search for any common factors.

L:=GF(3^8);
$L < x_1, x_2, x_3 >$:=PolynomialRing(L,3);
Factorization($D_1(x_1)$);
Factorization($D_1(x_2)$);
Factorization($D_1(x_3)$);

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