Universidade Federal Fluminense
Escola de Engenharia Industrial Metalúrgica de Volta Redonda
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Exercícios Resolvidos

Assunto: Integral Dupla

Comentários Iniciais:

É com imenso prazer que trago alguns exercícios resolvidos sobre integrais duplas e suas aplicações. Espero que você tenha um conspícuo aprendizado do tema. Não esqueça de constantemente recorrer aos livros, pois eles são excelente fonte de aprendizado.

Qualquer Dúvida me escreva.

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Reflexão

" Doce é a Luz e ver o sol deleita os olhos.
Se tu viveres por muitos anos, que os desfrute todos,
sempre lembrando que os dias sombrios são numerosos
e tudo o que acontece é vaidade. Estejas feliz na tua juventude
e afasta a tristeza do teu coração. Anda segundo os desejos
do teu coração, conforme o que teus olhos vêem.
Mas fica sabendo que por tudo o que fizeres aqui,
Deus te pedirá conta."

Salomão 935 a. C

1. Integral Dupla

$$\iint_R f(x,y)\,dxdy$$

$$\int f(x,y)\,dx \qquad y = cte$$

Exercícios Resolvidos

1.

$$\int_{y=0}^{2}\int_{y=0}^{x^2} y\,dydx$$

$$\frac{1}{2}\int_0^2 y^2 \Big|_0^{x^2} dx$$

$$\frac{1}{2}\int_0^2 x^4 dx$$

$$\frac{1}{2}\cdot\frac{1}{5}x^5 \Big|_0^2$$

$$\frac{1}{10}(2)^5 = \frac{32}{10} = \frac{16}{5}$$

2.

$$\int_0^1\int_0^2 (x+2)\,dydx$$

$$\int_0^1 (x+2)y \Big|_0^2$$

$$\int_0^1 (x+2)2dx$$

$$2\int_0^1 (x+2)dx$$

$$2\left[\frac{x^2}{2}+2x\right]\Big|_0^1$$

$$2\left(\frac{5}{2}\right) = 5$$

Outra forma:

$$\int_0^2\int_0^1 (x+2)\,dxdy$$

$$\int_0^2 \frac{x^2}{2}+2x \Big|_0^1 dx$$

$$\int_0^2 \frac{5}{2} dy$$

$$\frac{5}{2} y \Big|_0^2 = 5$$

Encontrou-se o mesmo resultado.

3.

$$\int_1^e \int_0^y \frac{1}{x^2 + y^2} dxdy$$

$$\int_1^e \frac{1}{y} arctg \frac{x}{y} \Big|_0^y dy$$

$$\int_1^e \frac{1}{y} (arctg1 - arctg0) dy$$

$$\int_1^e \frac{1}{y} \frac{\pi}{4} dy = \frac{\pi}{4} \int_1^e \frac{dy}{y} = \frac{\pi}{4} Lny \Big|_1^e$$

$$\frac{\pi}{4} [ln\, e - ln\, 1] = \frac{\pi}{4}$$

2. Interpretação da Integral Dupla

$$\iint_R f(x, y) dxdy$$

Seja $z = f(x, y)$ contínua na região R

$$V_i = f(x_i, y_j) \Delta x_i \Delta y_j$$

$$V \cong \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x_i \Delta y_j$$

$$V = \lim_{\substack{n \to \infty \\ m \to \infty}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta x_i \Delta y_j$$

$$V = \iint_R f(x, y) dxdy$$

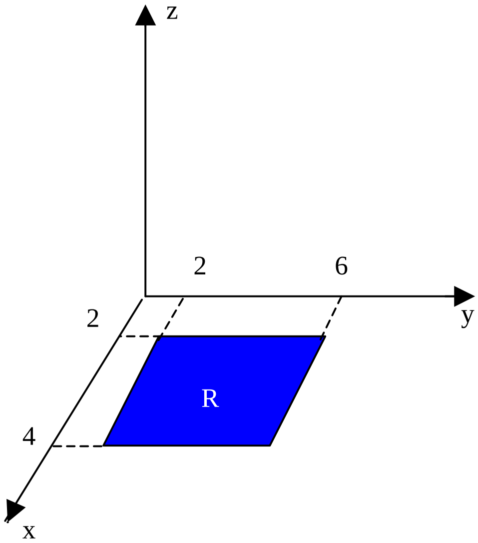
$$V = A \cdot b \cdot h \quad se \quad h = 1$$

$$V = A \cdot b$$

$$fazendo \quad f(x, y) = 1$$

$$\iint_R dxdy = A_R$$

1. Cálcule a área retangular R



$$A_R = \iint_R dxdy$$

$$R\begin{cases} 2 \le x \le 4 \\ 2 \le y \le 6 \end{cases}$$

$$A_R = \int_{x=2}^{4} \int_{y=2}^{6} dydx$$

$$A_R = \int_{x=2}^{4} y \Big|_{2}^{6} dx$$

$$A_R = \int_{x=2}^{4} 6 - 2dx$$

$$A_R = \int_{x=2}^{4} 4dx$$

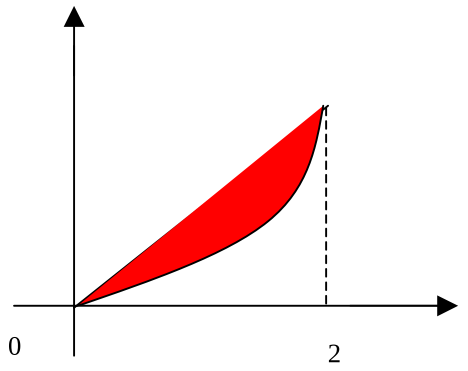
$$A_R = 4x \Big|_{2}^{4}$$

$$A_R = 16 - 8 = 8$$

3. Cálculo de áreas por Integral Dupla

$$A = \iint_R dxdy$$

1. Determinar a área da região limitada pelas curvas $y = x^3 e\ y = 4x$ no 1º Quadrante.



$$\begin{cases} y = x^3 \\ y = 4x \end{cases}$$

$$x^3 - 4x = 0 \begin{cases} 0 \\ +2 \\ -2 \end{cases}$$

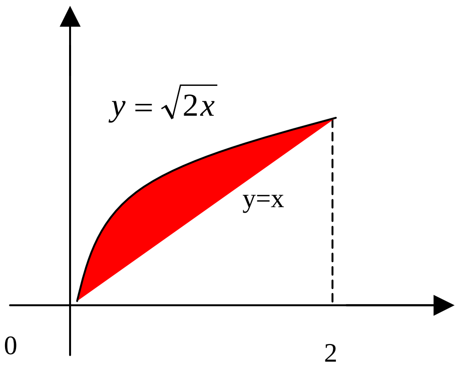
$$R = \begin{cases} 0 \le x \le 2 \\ x^3 \le y \le 4x \end{cases}$$

$$A = \int_{x=0}^{2} \int_{y=x^3}^{4x} dydx$$

$$A = \int_0^2 y \Big|_{x^3}^{4x} dx$$

$$A = \int_0^2 \left(4x - x^3\right) dx = 4\frac{x^2}{2} - \frac{x^4}{4}\Big|_0^2 = 4$$

2. Determinar a área da região limitada pelas curvas $y = \sqrt{2x}\ e\ y = x$ no 1º Quadrante.



$$y^2 = 2x \quad e \quad y = x$$

$$\begin{cases} y^2 = 2x \\ y = x \end{cases}$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\begin{cases} x = 0 \\ x = 2 \end{cases}$$

$$R\begin{cases} 0 \le x \le 2 \\ x \le y \le \sqrt{2x} \end{cases} ou \; R\begin{cases} 0 \le y \le 2 \\ \dfrac{y^2}{2} \le x \le y \end{cases}$$

$$A = \int_{x=0}^{2} \int_{y=x}^{\sqrt{2}x^{\frac{1}{2}}} dydx$$

$$A = \int_0^2 \left(\sqrt{2}x^{\frac{1}{2}} - x \right) dx$$

$$A = \frac{2}{3}\sqrt{2}(2)^{\frac{3}{2}} - \frac{x^2}{2} \Big|_0^2$$

$$A = \frac{2}{3}\sqrt{2}\left(\sqrt{2}\right)^3 - 2$$

$$A = \frac{8}{3} - 2 = \frac{8-6}{3} = \frac{2}{3}$$

Outra forma

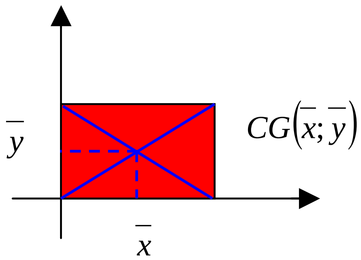
$$A = \int_{y=0}^{2} \int_{x=\frac{y^2}{2}}^{y} dxdy$$

$$A = \int_0^2 \left(y - \frac{y^2}{2} \right) dy$$

$$A = \frac{y^2}{2} - \frac{y^3}{6} \Big|_0^2$$

$$A = 2 - \frac{4}{3} = \frac{6-4}{3} = \frac{2}{3}$$

4. Momento e Centro de Gravidade de Áreas Planas

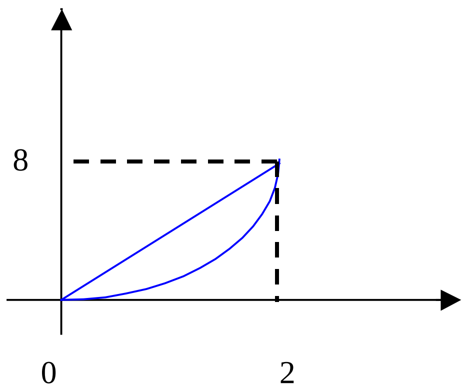


$$m_x = A \cdot \overline{y}$$
$$m_y = A \cdot \overline{x}$$

4.1. Coordenadas do Centro de Gravidade

$$\overline{x} = \frac{m_y}{A}$$
$$\overline{y} = \frac{m_x}{A}$$
$$A = \iint_R dydx$$
$$m_x = \iint_R ydydx$$
$$m_y = \iint_R xdydx$$

1. Determinar as coordenadas do centro de gravidade da Região limitada no 1º Quadrante por $y = x^3$ e $y = 4x$.



$$R = \begin{cases} 0 \le x \le 2 \\ x^3 \le y \le 4x \end{cases}$$

$$A = \int_{x=0}^{2} \int_{y=x^3}^{4x} dydx = 4$$

$$M_x = \int_{x=0}^{2} \int_{y=x^3}^{4x} ydydx$$

$$M_x = \frac{1}{2} \int_{x=0}^{2} y \Big|_{x^3}^{4x} dx$$

$$M_x = \frac{1}{2} \int_{x=0}^{2} \left(16x^2 - x^6\right) dx$$

$$M_x = \frac{1}{2} \left[\frac{16x^3}{3} - \frac{x^7}{7} \Big|_0^2 \right]$$

$$M_x = \frac{1}{2} \left[\frac{16 \cdot 8}{3} - \frac{128}{7} \right]$$

$$M_x = \frac{64}{3} - \frac{64}{7} = \frac{256}{21}$$

$$M_y = \int_{x=0}^{2} \int_{y=x^3}^{4x} xdydx$$

$$M_y = \int_{x=0}^{2} xy \Big|_{x^3}^{4x} dx$$

$$M_y = \int_{x=0}^{2} \left(4x^2 - x^4\right) dx$$

$$M_y = \frac{4x^3}{3} - \frac{x^5}{5} \Big|_0^2$$

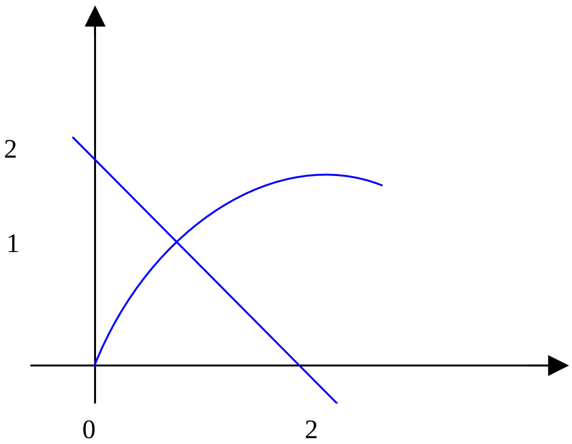
$$M_y = \frac{4 \cdot 8}{3} - \frac{32}{5} = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{64}{15 \cdot (4)} = \frac{16}{15}$$

$$\bar{y} = \frac{M_x}{A} = \frac{256}{21 \cdot (4)} = \frac{64}{21}$$

$$C.G. = \left(\frac{16}{15} ; \frac{64}{21} \right)$$

2. $y^2 = x; x + y = 2 \; e \; y = 0$ 1º Quadrante



$$R\begin{cases} 0 \le y \le 1 \\ y^2 \le x \le 2 - y \end{cases}$$

$$\begin{cases} y^2 = x \\ x + y = 2 \end{cases}$$

$$y^2 + y - 2 = 0 \begin{cases} S = -2 \\ P = 1 \end{cases}$$

$$A = \int_0^1 \int_{y^2}^{2-y} dxdy$$

$$A = \int_0^1 x \Big|_{y^2}^{2-y} dy$$

$$A = \int_0^1 \left(2 - y - y^2\right) dy$$

$$A = 2y - \frac{y^2}{2} - \frac{y^3}{3} \Big|_0^1$$

$$A = \frac{7}{6}$$

$$M_x = \int_0^1 \int_{y^2}^{2-y} ydxdy$$

$$M_x = \int_0^1 yx \Big|_{y^2}^{2-y} dy$$

$$M_x = \int_0^1 \left(2y - y^2 - y^3\right) dy$$

$$M_x = y^2 - \frac{y^3}{3} - \frac{y^4}{4} \Big|_0^1$$

$$M_x = \frac{5}{12}$$

$$M_y = \int_0^1 \int_{y^2}^{2-y} xdxdy$$

$$M_y = \int_0^1 \frac{x^2}{2} \Big|_{y^2}^{2-y} dy$$

$$M_y = \frac{1}{2}\int_0^1 \left(4 - 4y + y^2 - y^4\right)dy$$

$$M_y = \frac{1}{2}\left[4y - 2y^2 + \frac{y^3}{3} - \frac{y^5}{5}\Big|_0^1\right]$$

$$M_y = \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right]$$

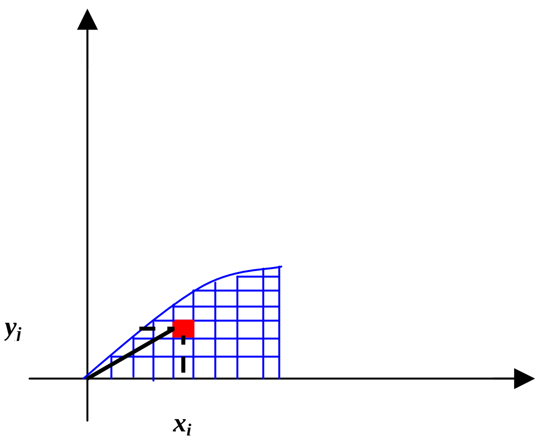
$$M_y = \frac{16}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{32}{35}$$

$$\bar{y} = \frac{M_x}{A} = \frac{5}{14}$$

$$C.G. = \left(\frac{32}{35}; \frac{5}{14}\right)$$

5. Momento de Inércia (I_x; I_y; I_0)

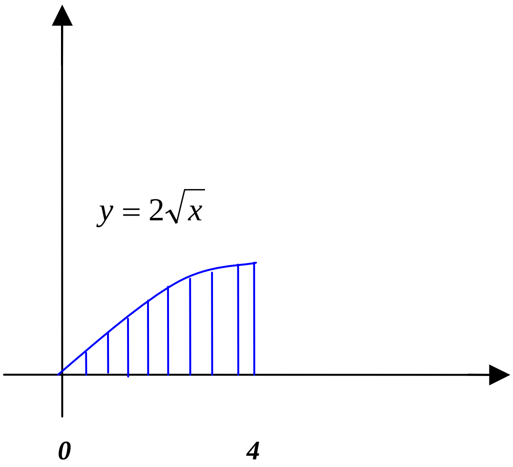


$$I_x = \iint_R y^2 dxdy$$

$$I_y = \iint_R x^2 dxdy$$

$$I_0 = I_x + I_y \quad ou \quad I_0 = \iint_R \left(x^2 + y^2\right)dxdy$$

1. Determinar os momentos de inércia $I_x; I_y \ e \ I_0$ da região limitada pelas curvas $y^2 = 4x;\ x = 4 \ e \ y = 0$ no 1º Quadrante.



$$R\begin{cases} 0 \le x \le 4 \\ 0 \le y \le 2\sqrt{x} \end{cases}$$

$$I_x = \int_0^4 \int_0^{2x^{\frac{1}{2}}} y^2 dy dx$$

$$I_x = \frac{1}{3}\int_0^4 y^3 \Big|_0^{2x^{\frac{1}{2}}} dx$$

$$I_x = \frac{1}{3}\int_0^4 8x^{\frac{3}{2}} dx$$

$$I_x = \frac{8}{3} x^{\frac{5}{2}} \frac{2}{5} \Big|_0^4$$

$$I_x = \frac{16}{15} \cdot 32 = \frac{512}{15}$$

$$I_y = \int_0^4 \int_0^{2x^{\frac{1}{2}}} x^2 dy dx$$

$$I_y = \int_0^4 x^2 y \Big|_0^{2x^{\frac{1}{2}}} dx$$

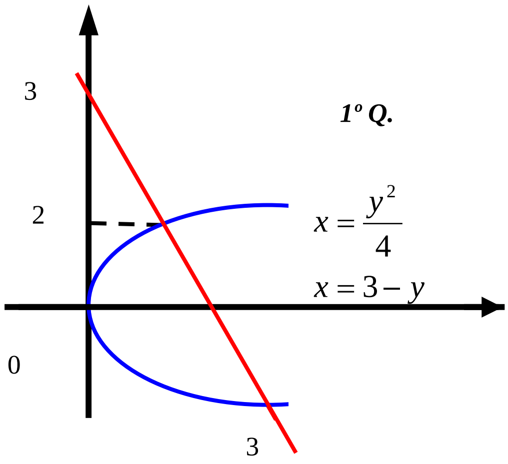
$$I_y = 2\int_0^4 x^{\frac{5}{2}} dx$$

$$I_y = \frac{4}{7} x^{\frac{7}{2}} \Big|_0^4$$

$$I_y = \frac{4}{7} \cdot 128 = \frac{512}{7}$$

$$I_0 = I_x + I_y = \frac{512}{15} + \frac{512}{7} = 107{,}28$$

2. $y^2 = 4x;\ x + y = 3;\ y = 0$



$$R\begin{cases} 0 \le y \le 2 \\ \frac{y^2}{4} \le x \le 3 - y \end{cases}$$

$$\begin{cases} y^2 = 4x \\ x + y = 3 \end{cases}$$

$$x = 3 - y$$

$$y^2 = 4(3 - y)$$

$$y^2 + 4y - 12 = 0\begin{cases} S = -4 \\ P = -12 \end{cases} \quad \begin{cases} y' = -6 \\ y'' = 2 \end{cases}$$

$$I_x = \int_0^2 \int_{\frac{y^2}{4}}^{3-y} y^2 dxdy$$

$$I_x = \frac{12}{5}$$

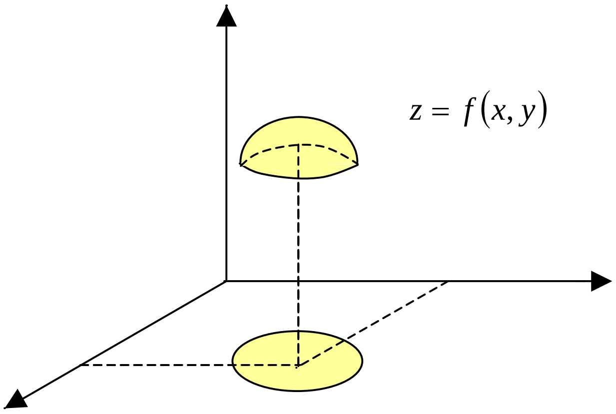
$$I_y = \int_0^2 \int_{\frac{y^2}{4}}^{3-y} x^2 dxdy$$

$$I_y = \frac{46}{7}$$

$$I_0 = I_x + I_y = \frac{12}{5} + \frac{46}{7} = 8{,}97$$

6. Volume por Integral Dupla

$z = f(x, y)$



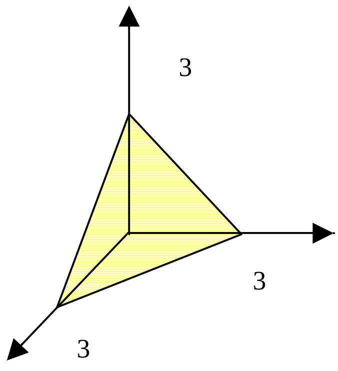
$$V_i = f(x_i, y_j)\Delta x \Delta y$$

$$\iint_R f(x,y)\,dxdy = V$$

$$V = \iint_R z\,dxdy$$

$$F(x,y,z)=0\begin{cases} z = f(x,y) \\ y = f(x,y) \\ x = f(x,y) \end{cases}$$

$x + y + z = 3$

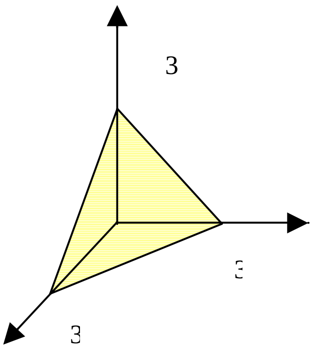


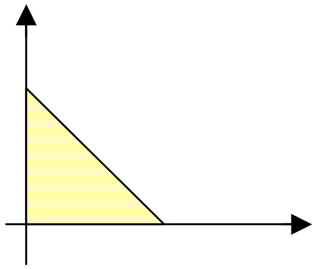
$z = 3 - x - y \qquad D \to plano\ xy$

$y = 3 - x - z \qquad D \to plano\ xz$

$x = 3 - y - z \qquad D \to plano\ yz$

1. Determinar o volume do Sólido limitado pelos planos coordenados pelo plano $x + y + z = 3$ no 1º octante.





$$Planos\ Coord.\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \qquad R\begin{cases} 0 \le x \le 3 \\ 0 \le y \le 3 - x \end{cases}$$

$$V = \int_{x=0}^{3}\int_{y=0}^{3-x}(3 - x - y)dydx$$

$$V = \int_0^3 3y - xy - \frac{y^2}{2}\Big|_0^{3-x} dx$$

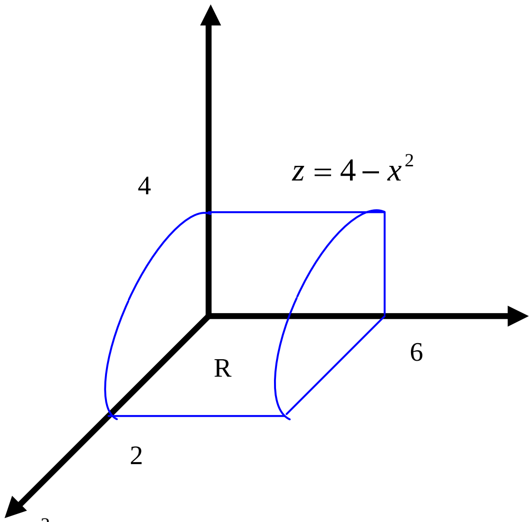
$$V = \int_0^3\left(\frac{9}{2} - 3x + \frac{x^2}{2}\right)dx$$

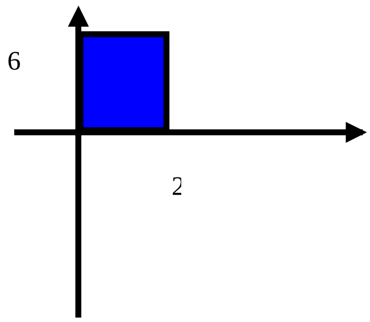
$$V = \frac{9}{2}x - \frac{3x^2}{2} + \frac{x^3}{6}\Big|_0^3$$

$$V = \frac{27}{2} - \frac{27}{2} + \frac{27}{6}$$

$$V = \frac{9}{2}u.v.$$

2. Determinar o volume do sólido limitado por $z = 4 - x^2; x = 0; y = 6; z = 0; y = 0$.





$$V = \int_0^2 \int_0^6 4 - x^2 \, dydx$$

$$V = \int_0^2 4y - x^2 y \Big|_0^6 dx$$

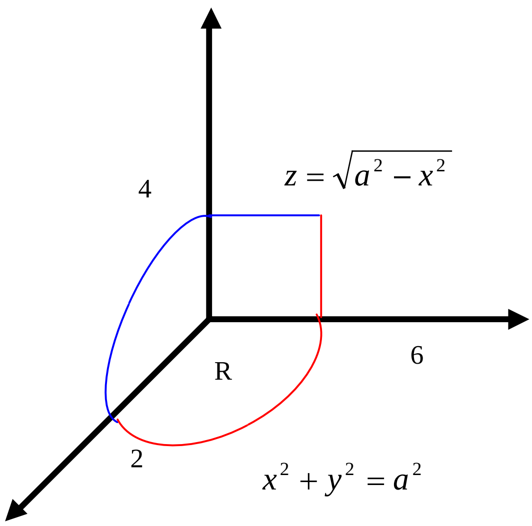
$$V = \int_0^2 24 - 6x^2 dx$$

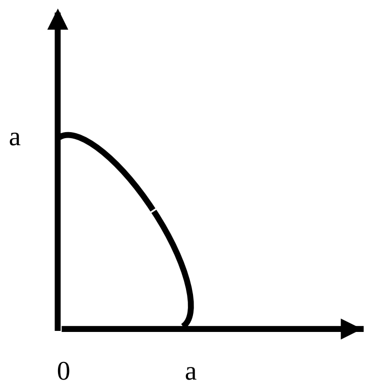
$$V = 24x - 2x^3 \Big|_0^6$$

$$V = 48 - 16$$

$$V = 32u.v.$$

3. Determinar o volume do sólido limitado no 1º octante pelos cilindros $x^2 + y^2 = a^2$ e $x^2 + z^2 = a^2$.





$$R = \begin{cases} 0 \le x \le a \\ 0 \le y \le \sqrt{a^2 - x^2} \end{cases}$$

$$V = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dydx$$

$$V = \int_{x=0}^{a} \sqrt{a^2 - x^2} \Big|_0^{\sqrt{a^2-x^2}} dx$$

$$V = \int_{x=0}^{a} \sqrt{a^2 - x^2} \cdot \sqrt{a^2 - x^2}\, dx$$

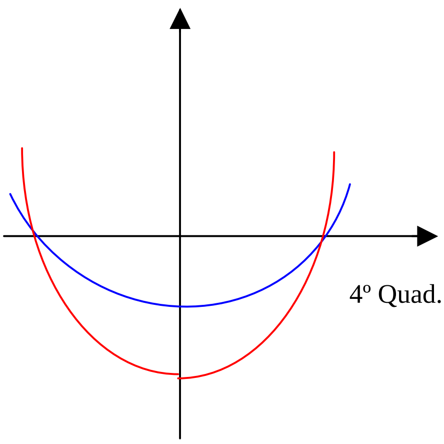
$$V = \int_{x=0}^{a} a^2 - x^2 dx$$

$$V = a^2 x - \frac{x^3}{3} \Big|_0^a$$

$$V = a^3 - \frac{a^3}{3}$$

$$V = \frac{3a^3 - a^3}{3} = \frac{2a^3}{3} u.v.$$

4. Determinar o volume do sólido limitado superiormente por $z = 2x + y + 4$ e inferiormente por $z = -x - y + 2$ e lateralmente pela superfície definida pelo contorno da região D limitada pelas curvas $y = x^2 - 4 \; e \; y = \frac{x^2}{2} - 2$.



$$D \begin{cases} y = x^2 - 4 \\ y = \frac{x^2}{2} - 2 \end{cases}$$

$$R \begin{cases} 0 \le x \le 2 \\ x^2 - 4 \le y \le \frac{x^2}{2} - 2 \end{cases}$$

$$z_1 = 2x + y + 4$$

$$z_2 = -x - y + 2$$

$$V = \int_0^2 \int_{x^2-4}^{\frac{x^2}{2}-2} (z_1 - z_2) dydx$$

$$V = \int_0^2 \int_{x^2-4}^{\frac{x^2}{2}-2} (3x + 2y + 2) dydx$$

$$V = \int_0^2 \int_{x^2-4}^{\frac{x^2}{2}-2} (3x + 2y + 2) dydx$$

$$V = \int_0^2 \left(3xy + y^2 + 2y\right) \Big|_{x^2-4}^{\frac{x^2}{2}-2} dx$$

$$V = \int_0^2 -\frac{3x^4}{4} - \frac{3x^3}{2} + 5x^2 + 6x - 8dx$$

$$V = -\frac{3x^5}{5} - \frac{3x^4}{8} + \frac{5x^3}{3} + 3x^2 - 8x \Big|_0^2$$

$$V = \frac{96}{5} - 6 + \frac{40}{3} + 12 - 16$$

$$V = \frac{338}{15} u.v.$$