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Interfaces with Other Disciplines

About negative efficiencies in Cross Evaluation BCC input oriented models

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ABSTRACT

It will be shown in this paper that the input oriented DEA BCC model can generate negative efficiencies that are usually hidden in the model. The impact of these negative efficiencies becomes obvious when using input oriented Cross Evaluation models. With the help of an example with one input and one output, the conditions for the possible occurrence of negative efficiencies will be shown. Furthermore, we will show that a small intuitive change in the BCC multipliers model, previously presented in other papers, corrects this situation. We show why this change is used and compared it with an alternative formulation, which avoid negative efficiencies, namely the Non-Decreasing Returns to Scale (NDRS) model. We also show that the formulation studied in this paper is less restrictive than the NDRS model. The study of this variation in the DEA BCC model will be complemented with the formulation of the dual envelope model. This model changes the original frontier. Using the concept of non-observed DMUs, those variations can be graphically analyzed. We have also carried out some algebraic studies concerning benchmarks, multipliers and returns to scale.

1. Introduction

The BCC DEA model was proposed by Banker et al. (1984) and introduces the hypothesis of variable returns to scale. Depending on whether the right orientation is chosen, the invariability of some translations, which this model includes, allows negative data to be used (Pastor, 1996; Thrall, 1996).

Although the BCC model with the adequate translation can use negative data, the efficiencies obtained by DEA models have always been taken as non-negative. In fact, the input or output oriented CCR model and the output oriented BCC model do not generate negative efficiencies. The latter model includes a greater or equal type constraint in the multipliers formulation, which ensures non-negativity for all efficiencies, as demonstrated in Section 3.

For all classic CCR or BCC models efficiency is the objective function result. In the specific case of the radial input oriented model the efficiency lies in the interval $[0,1]$.

Now, in the case of the DEA-BCC model, the multipliers formulation was initially the dual of the envelope formulation. The envelope formulation has an equality type constraint that makes the Production Possibility Sets to be convex. As the envelope formulation has one equality type constraint, its dual, the multipliers formulation, has one unrestricted (free) variable. This means we cannot ensure the non-negativity of the efficiencies when the multipliers of a DMU are used to evaluate other DMUs. When working with the classic DEA models this is not a problem. However, in Cross Evaluation models (Sexton et al., 1986; Doyle and Green, 1994) negative efficiencies can be generated by BCC input oriented models.

To correct this situation, this paper revisits a small intuitive change to the BCC multiplier model previously used by Angulo-Meza et al. (2004) and Wu et al. (2009). These authors have just used the change with no further studies. We shall interpret and study in this paper the consequences of this change. It will be shown that Angulo-Meza et al. (2004) and Wu et al. (2009) change generates a major alteration in the dual envelope model, causing a frontier displacement. Therefore, it also changes some benchmarks, multipliers and assumptions of returns to scale. We will also show that the modified BCC model brings further change: the efficiency of some DMUs will depend on both efficient and inefficient DMUs. We shall further go onto show the conditions for the existence of the negative efficiencies. Finally we shall compare the BCC modified model with the Non-Decreasing Returns to Scale (NDRS) model (Charnes et al., 1994; Cooper et al., 2007).

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2. Existence of implicit negative efficiencies in the DEA BCC model

The DEA-BCC model was proposed for dealing with situations whereby the proportionality between inputs and outputs is not constant along the efficient frontier. This generates a variable returns to scale (VRS) frontier. The BCC model was originally developed by adding a convexity restriction to the CCR (Charnes et al., 1978) model envelope formulation. The frontier is piecewise linear and takes into account the different production scales, like increasing, constant and decreasing returns to scale. The input oriented BCC multipliers model – the dual of the envelope model, is presented in (1).

$$\begin{aligned} \text{Max} \quad & \frac{\sum_{r=1}^{s}\mu_{ro}y_{ro}+\mu_{*o}}{\sum_{i=1}^{m}v_{io}x_{io}} \\ \text{S.t.} \quad & \frac{\sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}}{\sum_{i=1}^{m}v_{io}x_{ij}} \leqslant 1, \quad j=1,\ldots,n \\ & v_{io},\ \mu_{ro} \geqslant 0,\ \mu_{*o} \in \Re \end{aligned} \tag{1}$$

As well known in DEA literature this model can be linearized as in model (2).

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o} \\ \text{S.t.} \quad & \sum_{i=1}^{m}v_{io}x_{ij}=1 \\ & \sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}-\sum_{i=1}^{m}v_{io}x_{ij} \leqslant 0, \quad j=1,\ldots,n \\ & v_{io},\ \mu_{ro} \geqslant 0,\ \mu_{*o} \in \Re \end{aligned} \tag{2}$$

In this and similar models, DMU *o* is the DMU being evaluated, i.e., the observed DMU; μ_{ro} are the multipliers of the outputs, v_{io} are the multipliers of the *inputs*, *and* μ_{ro} and v_{io} are the model decision variables, y_{rj} is the output *r* of DMU *j*, and x_{ij} is the *input i* of DMU *j*, with $j = 1,\ldots,n$. The variable μ_{*o} indicates where the observed DMU finds itself: whether in increasing, constant or decreasing returns to scale frontier regions.

It should be noted in model (1), that when DMU *o* multipliers are used to evaluate other DMUs, the expression $\sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}$ may be negative, if μ_{*o} is negative enough. This means that the DMU *j* efficiency can be negative if and when evaluated by DMU *o*'s multipliers. This may occur when the DMU *o* is operating under decreasing returns to scale. However, these negative efficiencies do not appear in DEA results except when Cross Evaluation models are used. Even in this case, the negative efficiencies exist only in the input oriented model.

These negative efficiencies were previously detected by Soares de Mello et al. (2002), when using a smoothed DEA frontier, generalized by Nacif et al. (2009), as they carried out an input oriented Cross Evaluation with variable returns to scale. That work did not study the negative efficiencies themselves. To perform Cross Evaluation, Soares de Mello et al. (2002) have used an unsophisticated approach: DMUs that generated negative efficiencies in the Cross Evaluation model were not taken into account. Appa et al. (2006) have also noticed the negative efficiencies problem in BCC Cross Evaluation models.

Angulo-Meza et al. (2004) and Wu et al. (2009) have independently approached how to deal with negative efficiencies in the input oriented Cross Evaluation BCC model. Without further analyses, those authors limited themselves to add a set of constraints in the multipliers model to avoid negative efficiencies. The required interpretation and the consequences of the modified BCC model as well as the conditions for the existence of those negative efficiencies have yet to be studied and are the main object of this study.

In the case of just one output, $r = 1$, negative efficiencies may appear in the restrictions of model (1) for a specific DMU *j* when $\mu_{1o}y_{1j}+\mu_{*o}<0$, i.e., $y_{1j}<-\mu_{*o}/\mu_{1o}$.

Table 1 shows both data and results of a numerical example with one input and one output, illustrating the situation previously described. The multipliers were just the first obtained by the SIAD software (Angulo-Meza et al., 2005a,b). Notice that in Table 1, DMU A μ_{*o}, DMU F μ_{*f}, and DMU G μ_{*g} are negative.

Table 2 shows DMUs' efficiency indexes calculated by using other DMU multipliers, an approach similar to Cross Evaluation (Sexton et al., 1986; Doyle and Green, 1994). In this table, the values of cell *i/j* (i, j = A,$\ldots$,G) are DMU *j* efficiencies as evaluated by DMU *i*.

Table 2 shows that when using DMU A's output multiplier, $\mu_A = 0.375$, and the independent term, $\mu_{*A} = -2.750$, three negative efficiencies are obtained. This happens for DMUs B, C and E. The outputs of these DMUs are less than 7.333 (result of $-\mu_{*A}/\mu_A$). These values are meaningless in the classic theory of efficiency. Besides, these implicit negative efficiencies are the main reason to use of Cross Evaluation almost exclusively with constant returns to scale (CRS) DEA models.

3. Study of the BCC model with non-negativity constraints

In order to avoid the negative efficiencies we shall remember that they are caused only by the possible negativity of the independent term μ_{*o}. Therefore, a straightforward approach would be to restrain μ_{*o} to be non-negative, i.e., $\mu_{*o} \geqslant 0$. A model with this restriction is the Non-Decreasing Returns to Scale (NDRS) model (Charnes et al., 1994; Cooper et al., 2007). However, we shall note that not all negative values for μ_{*o} lead to negative efficiencies. In fact, negative efficiencies only appear if the absolute value of negative μ_{*o} is larger than the weighted sum of the output, i.e., negative efficiencies appear if and only if $\mu_{*o}<-\sum_{r=1}^{s}\mu_{ro}y_{rj}$. So, we must only avoid this condition to happen. We must restrain μ_{*o} to be not less $-\sum_{r=1}^{s}\mu_{ro}y_{rj}$ i.e., $\mu_{*o} \geqslant -\sum_{r=1}^{s}\mu_{ro}y_{rj}$, or $\mu_{*o}+\sum_{r=1}^{s}\mu_{ro}y_{rj} \geqslant 0$. With this additional constraint model (2) becomes model (3).

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o} \\ \text{S.t.} \quad & \sum_{i=1}^{m}v_{io}x_{ij}=1 \\ & \sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}-\sum_{i=1}^{m}v_{io}x_{ij} \leqslant 0, \quad j=1,\ldots,n \\ & \sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o} \geqslant 0, \quad j=1,\ldots,n \\ & v_{io},\ \mu_{ro} \geqslant 0,\ \mu_{*o} \in \Re \end{aligned} \tag{3}$$

Table 1
Numerical data example – 1 input and 1 output.

DMU	Input	Output	Multipliers			Efficiency
			v	μ	μ_*	
A	4	10	0.250	0.375	−2.750	1.000
B	1	5	1.000	0.000	1.000	1.000
C	2	7	0.500	0.000	0.500	0.500
D	1	8	1.000	0.125	0.000	1.000
E	6	6	0.167	0.000	0.167	0.167
F	5	10	0.200	0.300	−2.200	0.800
G	3	9	0.333	0.500	−3.667	0.833

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Table 2
Cross efficiencies for the numerical example.

	A	B	C	D	E	F	G
A	1.0000	−3.5000	−0.2500	1.0000	−0.3333	0.8000	0.8333
B	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
C	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
D	0.3125	0.6250	0.4375	1.0000	0.1250	0.2500	0.3750
E	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
F	1.0000	−3.5000	−0.2500	1.0000	−0.3333	0.8000	0.8333
G	1.0008	−3.5045	−0.2508	1.0000	−0.3338	0.8006	0.8338

In model (3), each restriction $\sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}\geqslant 0$ would be called a non-negativity restriction. This model is the linearized form of the nonlinear problem postulated by Wu et al. (2009) and Angulo-Meza et al. (2004), presented in model (4).

$$\begin{aligned} \text{Max} \quad & \frac{\sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}}{\sum_{i=1}^{m}v_{io}x_{ij}} \\ \text{S.t.} \quad & 0\leqslant\frac{\sum_{r=1}^{s}\mu_{ro}y_{rj}+\mu_{*o}}{\sum_{i=1}^{m}v_{io}x_{ij}}\leqslant 1, \quad j=1,\ldots,n \\ & v_{io},\ \mu_{ro}\geqslant 0,\ \mu_{*o}\in\Re \end{aligned} \tag{4}$$

We point out that all the denominators in model (4) are positive, due to the fact that we are dealing only with non-negative inputs and outputs. As the NDRS imposes μ_* to be greater or equal to zero and the non-negative efficiency constraint model is less restrictive than the NDRS model, as it only imposes μ_* to be greater or equal than minus the outputs weighted sum.For the output oriented BCC model there is also a free variable v_{*o}, and the nonlinear form of this model is presented in (5).

$$\begin{aligned} \text{Min} \quad & \frac{\sum_{i=1}^{m}v_{io}x_{ij}+v_{*o}}{\sum_{r=1}^{s}\mu_{ro}y_{rj}} \\ \text{S.t.} \quad & \frac{\sum_{i=1}^{m}v_{io}x_{ij}+v_{*o}}{\sum_{r=1}^{s}\mu_{ro}y_{rj}}\geqslant 1, \quad j=1,\ldots,n \\ & v_{io},\ \mu_{ro}\geqslant 0,\ v_{*o}\in\Re \end{aligned} \tag{5}$$

To avoid negative efficiencies as in model (4), we should add the constraint in (6).

$$\sum_{i=1}^{m}v_{io}x_{ij}+v_{*o}\geqslant 0 \tag{6}$$

The restriction in model (5) can be rewritten as

$$\sum_{i=1}^{m}v_{io}x_{ij}+v_{*o}\geqslant\sum_{r=1}^{s}\mu_{ro}y_{rj} \tag{7}$$

And, as $\mu_{ro},\ y_{rj}\geqslant 0$, then $\sum_{r=1}^{s}\mu_{ro}y_{rj}\geqslant 0$. So restriction (7) becomes restriction (6), i.e., restriction (6) is already implicitly included in model (5). Therefore, in output oriented BCC models there are no implicitly negative efficiencies, even in Cross Evaluation. This fact was used by Lins et al. (2003) when performing a model similar to Cross Evaluation, output oriented, to analyse Olympic results.

Table 3
Numerical example for the modified DEA-BCC model.

DMU	Input	Output	Multipliers			Efficiency
			V	μ	μ_*	
A	4	10	0.250	0.083333	−0.416667	0.416667
B	1	5	1.000	0.000	1.000	1.000
C	2	7	0.500	0.000	0.500	0.500
D	1	8	1.000	0.000	1.000	1.000
E	6	6	0.1666667	0.000	0.1666667	0.1666667
F	5	10	0.2	0.066667	−0.333333	0.333333
G	3	9	0.333333	0.111111	−0.555556	0.444444

Table 4
Cross efficiencies for the modified DEA BCC model.

	A	B	C	D	E	F	G
A	0.4167	0.0000	0.3333	1.0000	0.0556	0.3333	0.4445
B	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
C	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
D	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
E	0.2500	1.0000	0.5000	1.0000	0.1667	0.2000	0.3333
F	0.4167	0.0000	0.3333	1.0000	0.0556	0.3333	0.4444
G	0.4167	0.0000	0.3333	1.0000	0.0556	0.3333	0.4444
e_k	0.3214	0.5714	0.4286	1.0000	0.1190	0.2571	0.3810

Table 3 shows the results of model (3) using Table 1 data. The results show that the modified input oriented BCC model has decreased DMU A, F and G efficiencies as they are the only ones that generated negative efficiencies when used to evaluate other DMUs. For instance, the term $-\mu_{*A}/\mu_A$ of DMU A is equal to 5.00, being the lowest output value in the set of DMUs.

Once again we have used the first multipliers obtained with model (3) to calculate the Cross Evaluation Matrix for all DMUs. The results are shown in Table 4.

We draw the readers' attention to the fact that all evaluations carried out by all DMUs generate efficiencies between 0 and 1. It should be noted that DMU B has zero efficiency, when evaluated by A, F and G DMUs. In other words, the non-negative restriction is an active restriction. We may point out that despite the fact negative efficiencies are unacceptable in standard DEA model, nil value efficiencies may occur in those models but only when all outputs are zero. If we want to avoid also null efficiencies it is sufficient to replace zero by a non-Archimedean number, ε, in the non-negativity restriction in model (3), as is done in the non-Archimedean DEA models presented by Charnes et al. (1985).

In some cases, the modified DEA BCC model will not change any efficiency index. This is caused by the multiplicity of the optimum multipliers set. This situation is exemplified in Tables 5 and 6, with a data set of two inputs and one output.

Table 5
Data and BCC results for a three-dimensional numerical example.

DMU	Input1	Input2	Output1	Multipliers				Efficiency
				v_1	v_2	μ	μ_{*o}	
A	0.489	0.637	0.607	0.300	1.340	1.627	0.012	1.000
B	1.000	1.000	1.000	1.000	0.000	1.300	−0.300	1.000
C	0.019	0.190	0.010	30.702	2.193	0.000	1.000	1.000
D	0.032	0.008	0.005	30.702	2.193	0.000	1.000	1.000
E	0.096	0.052	0.032	0.000	19.231	19.173	0.058	0.672
F	0.053	0.035	0.007	17.754	1.687	15.240	0.505	0.612
G	0.898	0.164	0.115	0.000	6.098	6.079	0.018	0.717

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Table 6
Data and modified BCC results for a three-dimensional numerical example.

DMU	Input1	Input2	Output1	Multipliers				Efficiency
				v_1	v_2	μ	μ_{*o}	
A	0.489	0.637	0.607	0.300	1.340	1.627	0.012	1.000
B	1.000	1.000	1.000	0.183	0.817	0.993	0.007	1.000
C	0.019	0.190	0.010	26.989	2.564	23.168	0.768	1.000
D	0.032	0.008	0.005	30.525	2.900	26.203	0.869	1.000
E	0.096	0.052	0.032	0.000	19.231	19.173	0.058	0.672
F	0.053	0.035	0.007	17.754	1.687	15.240	0.505	0.612
G	0.898	0.164	0.115	0.000	6.098	6.079	0.018	0.717

These tables show that the classic BCC model (Table 5) and the modified BCC model (Table 6) provide the same efficiencies indexes for all DMUs even with a different set of multipliers.

Soares de Mello et al. (2002) have used a different approach to deal with negative efficiencies. In their paper they did not allow the DMUs generating negative efficiencies to evaluate other DMUs. In Table 2, rows corresponding to DMU A, F and G are not taken into account when calculating the average cross efficiency index, e_k for all and each DMU. Table 7 compares the rankings obtained by both approaches. In this case, rankings are the same.

4. Geometrical interpretation of the additional constraint

Discussing the effects of the new constraint on the efficient frontier is important. Thanassoulis and Allen (1988) have shown that multiplier restrictions can be replaced by one or more unobserved or artificial DMUs, i.e., DMUs that do not exist in the original data set. As the restrictions of non-negativity, one for each DMU, are in fact multiplier restrictions, they can be replaced by unobserved DMUs. So, the efficient frontier in the model with non-negativity constraints may also be dependent on the inefficient DMUs. The only inefficient DMUs that can change the frontier have, at least, one cross efficiency with negative value in the classic input oriented BCC model. Models with non-observed DMUs have been used by Thanassoulis et al. (2011), Jahanshahloo and Soleimani-Damaneh (2005), Sowlati and Paradi (2004), Diallo et al. (2007), Figueiredo and Soares de Mello (2009), Gonçalves et al. (2004), among others.

DMU A is the only efficient DMU, the efficiency value of which was changed by model (3) as shown by the results of the numerical two-dimensional example (Table 1). Thus, a single non-observed DMU is enough to modify the model frontier. The values of this non-observed DMU and that of DMU A outputs are the same. The non-observed DMU input is obtained by multiplying DMU A efficiency in the modified model by the actual input of DMU A. The non-observed DMU output and input are respectively 10 and 1.6667 for our numerical example.

Fig. 1 shows the frontiers of the DEA BCC, the modified DEA BCC and the NDRS models. In this figure, the black line represents the three models common frontier; the continuous line in red belongs only to the DEA BCC model; the red discontinuous line represents the frontiers of the modified DEA BCC model, and the blue line represents the NDRS frontier. The modified DEA BCC frontier displacement from the original DEA BCC frontier is clearly seen. The modified DEA BCC frontier is closer than the NDRS one to the original frontier.

Table 7
Average cross efficiency indexes and ranking for both approaches.

Ranking	Approaches	
	Non-negativity constraint	Soares de Mello et al. (2002)
D	1.000	1.000
B	0.5714	0.9063
C	0.4286	0.4844
G	0.3810	0.3438
A	0.3214	0.2656
F	0.2571	0.2125
E	0.1190	0.1563

Besides the changes in the efficient frontier, other changes may occur when introducing the non-negativity constraint. One of those changes is in the scale elasticity. As the modified model is closer to CCR model than the original BCC model, scale elasticity presents less variation than in the BCC model. Since Forsund and Hjalmarsson (2004) and Cooper et al. (2006) shown that the scale elasticity is a function of μ_{*o}, we will analyze that variable instead of analyzing the scale elasticity itself. As mentioned before, these values of μ_{*o}, as well as the values of all multipliers, are note uniquely determined. So, instead of a single value of μ_{*o}, we determine its range both in the BCC and in the modified BCC models. To determine the range of μ_{*o}, for each model, we have used the efficiency indexes obtained by each model (Table 1 for the classic BCC model and Table 3 for the modified BCC model). Then, we have obtained the maximum value of μ_{*o} for each DMU with a new model whose goal is to maximize the value of μ_{*o}, with the constraints in the previous model and maintaining the efficiency index (a new restriction) found previously. The same was done to obtain the minimum value of μ_{*o}. The formulation for these BCC model is presented in (8). This was done for both models. Table 8 shows the results for the μ_{*o} range in both models.

$$\begin{aligned} \text{Max (or Min)} \quad & \mu_{*o} \\ \text{S.t.} \quad & \sum_{i=1}^{m} v_{io} x_{ij} = 1 \\ & \sum_{r=1}^{s} \mu_{ro} y_{rj} + \mu_{*o} - \sum_{i=1}^{m} v_{io} x_{ij} \leqslant 0, \\ & j = 1, \ldots, n, \text{ and } j \neq 0 \\ & \sum_{r=1}^{s} \mu_{ro} y_{ro} + \mu_{*o} - Eff_o \sum_{i=1}^{m} v_{io} x_{i0} = 0 \\ & v_{io}, \ \mu_{ro} \geqslant 0, \ \mu_{*o} \in \Re \end{aligned} \tag{8}$$

All DMUs shown in this table that have changed their efficiency in the modified BCC model have new values for the μ_{*o} range. On top of it, DMU D, that did not change its efficiency in the modified BCC model has also a changed value for the μ_{*o} lower bound. This happens because this DMU is located at the point where the two frontiers separate from each other.

5. Interpretation of the new constraint using the envelope model

The geometric representation in Fig. 1 is only valid for one input and one output. The multidimensional case can only be interpreted

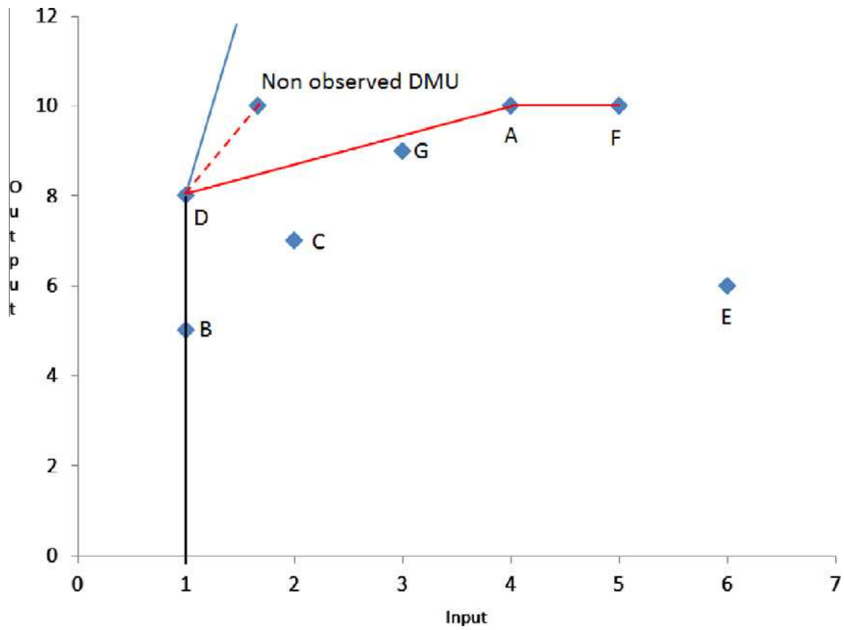

Fig. 1. The BCC, the modified BCC and the NDRS frontiers.

Table 8
Ranges for μ_{*o} in classic BCC and modified BCC models.

DMU	Classic BCC μ_{*o}		Modified BCC μ_{*o}	
	Max	Min	Max	Min
A	−2.7500	Unbound	−0.4167	−0.4167
B	1.0000	1.0000	1.0000	1.0000
C	0.5000	0.5000	0.5000	0.5000
D	1.0000	−11.0000	1.0000	−1.6667
E	0.1667	0.1667	0.1667	0.1667
F	−2.2000	Unbound	−0.3333	−0.3333
G	−3.6640	−3.6640	−0.5556	0.5556

using the model (3) dual, i.e., the envelope formulation for the modified BCC model. This formulation is presented in model (9).

$$
\begin{aligned}
\text{Min}\quad & \theta_o \\
\text{S.t.}\quad & \sum_{j=1}^{n} y_{rj}\lambda_j - \sum_{j=1}^{n} y_{rj}\lambda'_j \geqslant y_{ro}, \quad \forall r \\
& \sum_{j=1}^{n} x_{ij}\lambda_j \leqslant \theta x_{io}, \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n} \lambda'_j = 1, \quad \forall j \\
& \lambda_j,\ \lambda'_j \geqslant 0,\ \forall j
\end{aligned}
\tag{9}
$$

Model (3) additional constraints, one for each DMU, generate the same number of additional decision variables in the dual model (9). These variables are called λ'. It should be noted that when the sum of all λ' is not null, then the sum of λ will not be unitary. Therefore, in this case, there will be no guarantee of convexity in the modified model.

Model (9) also determines benchmarks for the DMUs' set. Table 9, shows the values of λ and λ'. All benchmarks are real DMUs. Although a geometrical observation of Fig. 1 may lead to the conclusion that an unobserved DMU is a benchmark for G, we can see in Table 9, the unobserved DMU is not a benchmark. In fact, the unobserved DMU is not a real DMU, it was not included in the data set, and consequently, it cannot be a benchmark for any inefficient DMU. The unobserved DMU is only a tool to enable us to visualize the new shape of the efficient frontier.

Table 9
Values of λ and λ'.

DMU	λ	λ'
A	$\lambda_D = 1.666667$	$\lambda'_B = 0.666667$
B	$\lambda_B = 1$	–
C	$\lambda_B = 0.333333$, $\lambda_D = 0.6666667$	–
D	$\lambda_D = 1$	–
E	$\lambda_B = 666667$, $\lambda_D = 0.3333333$	–
F	$\lambda_D = 1.666667$	$\lambda'_B = 0.666667$
G	$\lambda_D = 1.333333$	$\lambda'_B = 0.333333$

For the DMUs that have DMU A as one of the benchmarks in the classic BCC model the new benchmark is DMU D alone. For the other DMUs there is no change in the benchmarks' set.

From the complementary slack theorem, we know that λ' times the corresponding constraints slack must be null. Consequently, the necessary, but not sufficient, condition for λ' to be other than zero, is for the slack in the corresponding additional restriction of model (9) to be zero. This happens when the corresponding additional restriction is active. This means the DMU corresponding to the active restriction will have negative efficiency using DMU o multipliers.

In the numerical example showed in Table 3, DMU A efficiency has been changed owing to the additional constraints, and this change was caused by the non-negativity constraint relative to DMU B. So, all DMU A λ', except λ'_B, are null. Similar considerations can be made to DMUs F and G.

6. Final comments

This paper has shown that the input oriented DEA BCC model can generate negative efficiencies when using Cross Evaluation models. The one input and one output example has defined the one condition for the possible occurrence of negative efficiencies.

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Negative efficiencies are avoided with the non-negative constraint presented by Angulo-Meza et al. (2004) and Wu et al. (2009). We have shown that this constraint is less restrictive than some alternative approaches, namely the NDRS model.

As the model with the non-negativity constrain is closer to the original BCC model, a Cross Evaluation with an input oriented BCC model should use this restriction instead of the NDRS model. Of course, the average cross efficiency scores are higher using the non-negativity restriction and lower using the NDRS. We shall point out that the model with the non-negativity restriction is not a pure BCC model, and so further research is needed in the matter of BCC Cross Evaluation. For instance, using the virtual peer approach described in Rödder and Reucher (2012).

Also, the inclusion of a new set of constraints in the BCC multipliers model generates an important modification in the dual (envelope) model. The new dual model, with a new set of variables, can change the efficient frontier.

This paper includes two numerical examples. The first one has shown that the modified model causes a change in the efficient frontier. These changes have been analyzed through the introduction of a non-observed DMU. In this first example, the DMU causing negative efficiencies in other DMUs has had an efficiency decrease in the modified model. This decrease did not occur in the second example (two inputs and one output). This has happened on account of the optimum multiplier multiplicity.

We have also analyzed the algebraic characteristics of the modified BCC model, and have empirically observed the relationship between λ and λ'. In all cases, the benchmarks were always real DMUs.

Possible changes in the efficient frontier, and consequently in the efficiency indexes, may occur only in DMUs that generate negative efficiencies when analyzing other DMUs. However, changes in the range of multipliers and in the scale factor may occur in DMUs whose efficiency is not affected by the new constraint. This happens to DMUs located in the points where the BCC and the modified BCC frontiers separate from each other. It is obvious that the model presented in this paper is a new DEA model different from the BCC model and with different assumptions. One of its many characteristics is that the location of the unobserved DMU depends on some efficient and inefficient DMUs. In other words, contrary to standard DEA model, in the model presented here the frontier shape depends also on some inefficient DMUs.

It should be noted that the negative efficiencies mentioned in this work apply neither to the CCR model nor to the output oriented BCC model. The latter includes a greater or equal type constraint in the multipliers model, ensuring the non-negativity of efficiency measures, as demonstrated in Section 3.

It is important to observe the model presented here solves one of the problems of using Cross Evaluation with the variable returns to scale models. Other problems may occur and will be studied in future works. Also, other approaches may exist to deal with negative efficiencies in Cross Evaluation.

We may point out that this study has been developed for non-negative data. Further studies are required regarding the presence of negative data in the modified BCC model with non-negativity constraints.

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