Paper published in EJOR. DOI: 10.1016/j.ejor.2016.05.012

A new approach to the bi-dimensional representation of the DEA efficient frontier with multiple inputs and outputs

Carlos A. Bana e Costa[a]

João Carlos C. B. Soares de Mello[b*]

Lidia Angulo Meza[b]

[a] CEG-IST, Centre for Management Studies of IST, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais, 1049-001, Lisboa, Portugal

[b] Department of Production Engineering (TEP), Universidade Federal Fluminense, Rua Passo da Pátria 156, São Domingos, 24210-240, Niterói, Rio de Janeiro, Brazil

*Corresponding author: +55 21 26295368, jcsmello@producao.uff.br, Rua Passo da Pátria, 156, São Domingos, Niterói, RJ, Brazil, 24210-240.

Abstract

This paper presents a new approach to the graphical presentation of DEA results. Whatever the number of inputs and outputs are, an adequate normalization of their weights is enough to generate a simple bi-dimensional graph, similar to that of the CCR frontier with one input and one output. An advantage over other approaches to the same representation problem is that no complementary techniques are required to plot the frontier. It is also proved that the distance of a DMU to the frontier is related to its efficiency. The proposed approach is also valid for the BCC model.

Keywords: Data Envelopment Analysis; Efficient Frontier; Graphical representation

1. Introduction

In Data Envelopment Analysis (DEA), graphical representations have been used since the seminal paper of Charnes et al (1978) to show the position of each DMU in relation to the efficient frontier. These representations are powerful support tools for decision makers, for instance to ascertain how far the DMUs are from the efficient frontier, or to look for concentrations of DMUs in some areas in the graph indicating concentration in a market sector.

Initially, bi-dimensional graphical representations in DEA were limited to cases of three variables at most, whether it was a one input-two outputs case or a two inputs-one output case. Later, some researchers looked into graphically representing results of DEA models with more than three variables. However, the techniques proposed so far entail some difficulties, like harder visualisation with the increase of variables, use of transformed DEA models which are difficult to interpret, and a lack of clear representation of the efficient frontier, among others, as highlighted in Section 3. In order to avoid these difficulties, this paper proposes a novel bi-dimensional representation of the DEA frontier and the DMUs' location in relation to this frontier, which extends and details the initial suggestion presented by Bana e Costa et al (2014).

In the next section, the standard graphical representations of the efficient frontiers and DMUs are presented. This includes the three variable representation of the CCR model. This is followed by a literature review of different graphical representations and comments in Section Three. Section Four introduces the proposed bi-dimensional graphical representation and in Section Five a numerical example is presented. The proposed approach is extended for the BCC model in Section Six. Section Seven illustrates the bi-dimensional graphical representation with real data. In Section Eight we addressed the issue of the multiple optimal weights for efficient DMUs. Finally, Section Nine outlines conclusions of the paper.

2. Standard bi-dimensional DEA representation

Graphical representations for efficiency analysis were used before DEA (Farrell, 1957) to represent the definition of technical efficiency or the theoretical production function or isoquant. When they introduced DEA, Charnes et al (1978) used two inputs and one output to show the efficient frontier and the location of the DMUs. This was done by transforming the three variables into only two variables, namely input 1 divided by the output and input 2 divided by the output. The CCR fractional model proposed by Charnes et al (1978), which assumes constant returns to scale (CRS), is presented in (1).

$$
\begin{aligned}
& Max \ \frac{\sum_r u_r y_{ro}}{\sum_i v_i x_{io}} \\
& s.t. \\
& \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \leq 1, \ \forall j \\
& u_r \geq 0, \ v_i \geq 0, \ \forall r, i
\end{aligned}
\qquad (1)
$$

In this model x_{io} is the input i of DMU o; y_{ro} is the output r of DMU o; v_i is the weight of the input i; u_r is the weight of the output r; x_{ij} is the input i and y_{rj} is the output r of DMU j. The objective function of this formulation is the efficiency index of DMU o, which is the ratio of the virtual output (the weighted sum of the outputs) by the virtual input (the weighted sum of the inputs) of DMU o. This model is run for every DMU in the set.

It should be noted that both the objective function and the constraints are fractional, so it is easy to see (Coelli et al., 1998, Cooper et al., 2007) that if a set of multipliers is an optimal solution for model (1) another solution is obtained with all the multipliers multiplied by the same strictly positive number. This means that model (1) has multiple optimal solutions for all DMUs. In order to obtain only one solution, Charnes et al (1978) introduced the constraint that the objective function denominator (the weighted sum of the inputs, i.e. the virtual input) equals 1. This is a key point since with such a constraint one also obtains the linearization of model (1). Thus the numerator (the weighted sum of the outputs, i.e. the virtual output) becomes the new objective function and it is equal to the DMU o efficiency index. This formulation is presented in model (2).

$$
\begin{aligned}
& Max \sum_r u_r y_{ro} \\
& s.t. \\
& \sum_i v_i x_{io} = 1 \qquad\qquad (2) \\
& \sum_r u_r y_{rj} - \sum_i v_i x_{ij} \leq 0,\ \forall j \\
& u_r \geq 0,\ v_i \geq 0,\ \forall r,i
\end{aligned}
$$

Model (2) provides the efficiency index of the observed DMU (DMU *o*) which is a real number between 0 and 1; as well as the inputs and outputs weights (also called multipliers) to obtain the value of the efficiency index. This formulation is frequently called the multiplier model. Its dual model, called the envelopment model, provides information about the benchmarks, i.e. the DMU *o* reference set, and the targets, i.e. inputs and outputs levels for DMU *o* to become efficient, and of course the efficiency index. The graphical representation of the efficient frontier and the DMUs' location for the one input and one output case is depicted in figure 1.

In figure 1 the efficient frontier is represented by a diagonal that begins at the origin and passes through the most productive DMU or DMUs; DMUs A and B in this example. All DMUs under this frontier are inefficient. Geometrically, the efficiency index is ascertained by measuring the distance of the DMU from the efficient frontier. In the input oriented model, DMU D's efficiency is $\overline{OA}/\overline{OD}$, which clearly indicates that the output levels are maintained while trying to reduce the inputs used. On the other hand, in the output oriented model, DMU D's efficiency is $\overline{O'D}/\overline{O'D'}$, which indicates that the input levels are maintained while trying to increase the outputs. These measures of efficiency were first devised by Farrell (1957) based on the works by Debreu (1951).

The model presented in (1) and linearized in (2) is input oriented, but in the case of CRS both the input oriented and output oriented efficiency indexes are the same. For details about the output oriented CCR model and respective multipliers and envelopment models, see Cooper et al (2004), for instance.

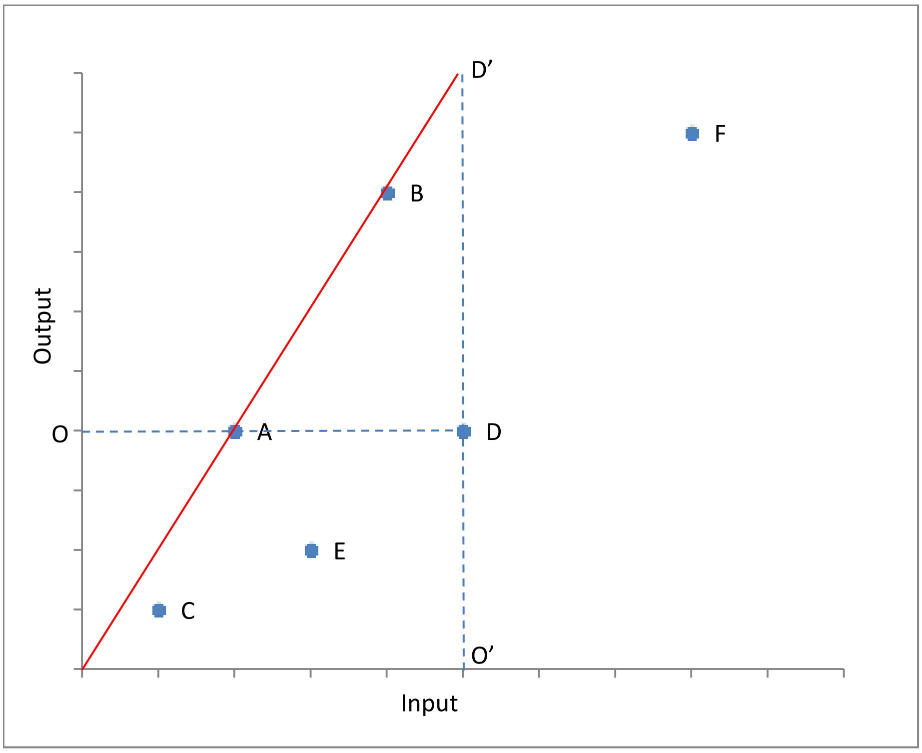

Figure 1. CCR efficient frontier for one input and one output.

This standard graphical representation for the CCR model in the plane can also be charted using three variables – two inputs and one output, or one input and two outputs. In the former case, each axis represents each input divided by the output; in the latter, each axis represents each output divided by the input. Figure 2 depicts the CRS frontier for one input and two outputs and also the projection of inefficient DMU F.

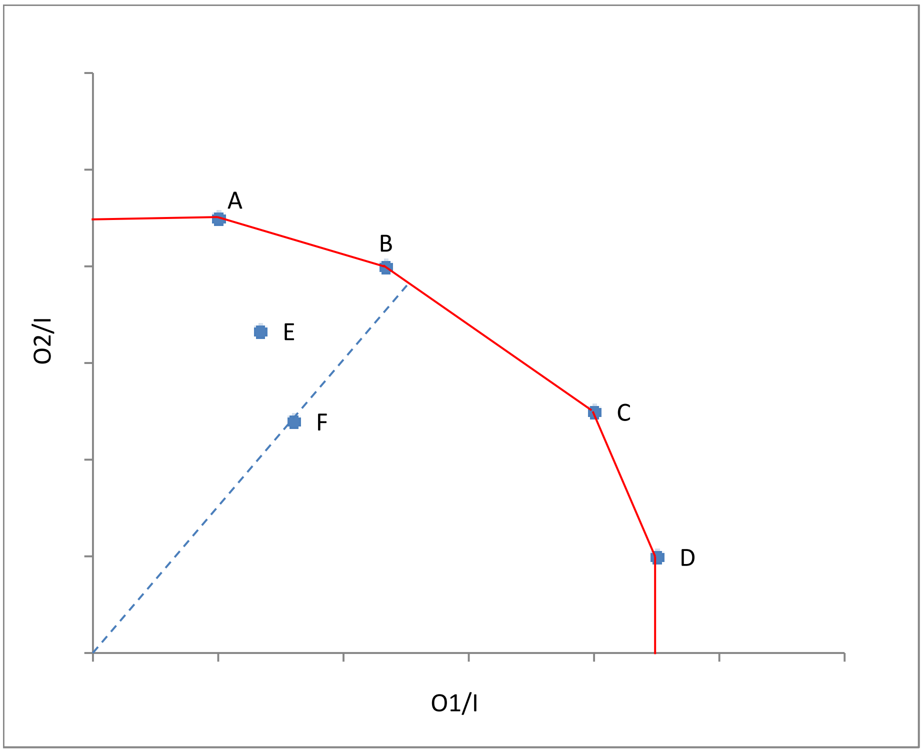

Figure 2. CRS efficient frontier for one input and two outputs.

Later, Banker et al (1984) also used a bi-dimensional representation, for one input and one output, to show a variable returns to scale (VRS) efficient frontier and the differences between the technical and scale efficiencies. The input oriented BCC model is presented in (3) and its linearized formulation is presented in (4).

$$
\begin{aligned}
& Max \ \frac{\sum_r u_r y_{ro} + u_*}{\sum_i v_i x_{io}} \\
& s.t. \\
& \frac{\sum_r u_r y_{rj} + u_*}{\sum_i v_i x_{ij}} \le 1, \ \forall j \qquad (3) \\
& u_r \ge 0, \ v_i \ge 0, \ \forall r,i \\
& u_* \in \Re
\end{aligned}
$$

$$\begin{aligned}
& Max \sum_r u_r y_{ro} + u_* \\
& s.t. \\
& \sum_i v_i x_{io} = 1 \\
& \sum_r u_r y_{rj} + u_* - \sum_i v_i x_{io} \leq 0, \ \forall j \\
& u_r \geq 0, \ v_i \geq 0, \ \forall r, i \\
& u_* \in \Re
\end{aligned} \tag{4}$$

The BCC model includes a new variable u_* that accounts for the scale where DMU *o* operates. The one input one output representation of the efficient frontier considering VRS is depicted in figure 3.

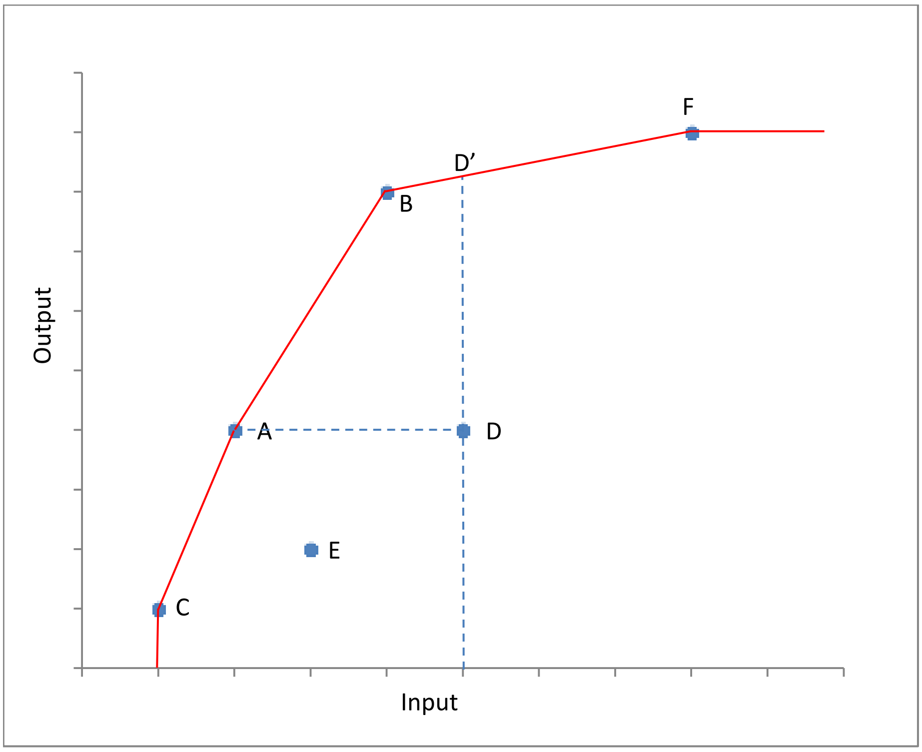

Figure 3. BCC efficient frontier for one input and one output.

In figure 3, both the input oriented and output oriented efficiency indexes are calculated as before, but the efficiency indexes for both orientations are not necessarily the same. Moreover, it will not be possible to include any other input or output in this graphical representation under the VRS assumptions.

3. Literature Review

In the previous section, the standard graphical representations of the efficient frontier and the DMUs' location in the plane were limited to three variables (two inputs-one output or one input-two outputs) for the CCR model and two variables (one input and one output) for the BCC model. However, some researchers have worked on providing alternative techniques to also represent the DEA frontier and/or the DMUs' location in the plane in the presence of more variables. To view other DEA results, for example charts of efficiency statistics, correlations, use of inputs and outputs, etc., see El-Mahgary and Lahdelma (1995), for instance.

Desai and Walters (1991) proposed a parallel axis representation for multiple variables and demonstrated how this representation may be used as a tool for implementing the DEA results. However, the increasing number of variables leads to more complicated visualisation. Moreover, there is no clear definition of the efficient frontier and the localization of DMUs.

Using a modified DEA model based on a multiple criteria value function, Belton and Vickers (1993) implemented a visual interactive decision support system called VIDEA (Visual Interactive Data Envelopment Analysis), an extension of VISA (Belton and Vickers, 1992). Compared to the standard DEA model, the modified DEA model has new restrictions, in that the total sum of the input weights equals 1, as well as the total sum of the output weights, and they proposed using a different objective function. The VIDEA system provides the modified weights profiles and an efficiency plot which shows the aggregate output on the Y axis versus the aggregate input on the X axis. In this plot the efficient DMUs are those that appear in the north-west frontier as in the BCC frontier for one input and one output shown in figure 3. The model also allows the effect of varying the weights to be investigated. Even though there is clear definition and visualisation of the efficient DMUs, there is no guarantee that the distance from a DMU to the frontier is related to its efficiency index. Moreover, the modified DEA model does not provide an efficiency index. Later, Stewart (1996) presented a slight variation of this formulation which provides an efficiency index, but no graphical representation of this new formulation was presented.

Hackman et al (1994) presented an algorithm to explicitly represent a two-dimensional section of a production possibility set for any number of inputs or outputs and also to determine the points in the graph. The resulting graph is a series of "frontiers" all depicted in the same quadrant. Later, Maital and Vaninsky (1999) extended this approach to gradually project an inefficient DMU onto the frontier. As expected, visualisation becomes harder as the number of DMUs increase.

A brief presentation and analysis of the aforementioned papers can be found in Adler and Raveh (2008), who used Co-Plot (Raveh, 2000) to present DEA graphically. This approach was later used by Huang and Liao (2012) in commercial banks. It allows graphical representation of any number of DMUs in the presence of multiple variables. However, even though these authors state that the efficient DMUs are located in the outer ring or sector of the plot, there is no clear representation of the efficient frontier.

When introducing a new non-radial efficiency projection with the so-called distance friction minimisation (DFM) method, Suzuki et al (2010) used a bi-dimensional weighted variable representation to depict the non-radial efficient projections. This representation is limited to two weighted inputs, or two weighted outputs.

Bougnol et al (2010) introduced a 3-D representation of the efficient frontier under CRS with three variables (two inputs and one output) and Ozcan et al (2010) presented IDEAL, a software that allows 3-D graphical representation under VRS. Visualisation, limited to three variables, becomes harder as the number of DMUs increase.

Later, Akçay et al (2012) developed the software SmartDEA to solve DEA models. The software presents the DEA results and some additional data. They proposed integrating DEA results with data mining and information visualisation. This software has two types of graphical presentations, coloured scatter plots and tile graphs. The first graphical representation, called “starfield” visualisation in information visualisation terminology, allows the representation of several variables of the data on a two dimensional space, and colours reflect the efficiency of the DMU or the values of another variable; in addition to that, the size of the DMUs can represent any other data. The other type of graphical presentation comprises tile graphs, which divide a bounded surface into rectangular regions, clustering data in line with a chosen criterion (DEA variables, other data or DEA model results). In each cluster, each element receives the appropriate area according to a numerical variable. Although many data and DEA results can be represented using this software, it does not provide an efficient frontier representation.

Appa et al (2010), in a case study of the efficiency analysis of electricity distributors in Brazil using a one-input and four-outputs DEA model, presented a bi-dimensional graphical representation where the x-axis depicted the input and the y-axis represented the virtual output, that is, the weighted sum of the outputs. Such graphical representation was possible due to the use of a modified CCR model, based on Thanassoulis (2000)'s work. In the graphical representation of Appa et al (2010) a non-standard normalization was used to linearize the CCR fractional formulation presented in (1). A clear definition of the efficient frontier and the localization of the DMUs are the advantages of this representation; however, it is limited to one input and multiple outputs.

Two of the aforementioned graphical representations are closely related to the approach presented in this paper. Belton and Vickers (1993), on the one hand, and Appa et al (2010), on the other hand. Both graphical approaches use a modified DEA model. The Appa et al (2010) model was restricted to deal with one single input and multiple outputs; the approach in this paper may be considered a multidimensional generalisation of that approach. The modified DEA model used by Belton and Vickers (1993) to develop their graphical representation does not provide an efficiency index. Moreover, using their model the distance from the DMU to the frontier in the two dimensional graphical representation is not related to the standard DEA efficiency.

4. The proposed multiple-inputs and multiple-outputs DEA two-dimensional representation

4.1. The input oriented case

In order to represent all DMUs in a bi-dimensional plot, the virtual input and the virtual output may be used. The issue is that in the standard input oriented CCR (and BCC) model the restriction added to linearize the formulation states that the virtual input equals 1 (as in model (2)). So, in a virtual input versus virtual output plot all DMUs would be located on the same vertical straight line. Such a graphical representation would be meaningless. This leads to the need for a different constraint to avoid the existence of multiple optimal solutions for model (1). Instead of the usual normalization, a restriction that states that the total sum of the input weights equals 1 is introduced. This turns the input oriented CCR model in (1) into model (5).

$$
\begin{aligned}
& Max\ \frac{\sum_r u_r y_{ro}}{\sum_i v_i x_{io}} \\
& s.t. \\
& \sum_i v_i = 1 \\
& \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \le 1,\ \forall j \\
& u_r \ge 0,\ v_i \ge 0,\ \forall r, i
\end{aligned} \tag{5}
$$

The resulting formulation (5) is still a fractional program. However, with a simple mathematical operation, we can use the results of the linearized CCR model (2) by dividing its resulting weights by the total sum of the input weights of the DMU *o*. Let S_j be the total sum of the input weights of DMU *j*, as in (6)

$$S_j = \sum_i v_{ij} \tag{6}$$

where v_{ij} is the weight of the input *i* for DMU *j*. Let v'_{ij} be the modified input weight of the input *i* for DMU *j* and u'_{rj} be the modified output weight of the output *r* for DMU *j* as in (7) and (8).

$$v'_{ij} = \frac{v_{ij}}{S_j} \tag{7}$$

$$u'_{rj} = \frac{u_{rj}}{S_j} \tag{8}$$

And, let I'_j be the modified virtual input and O'_j be the modified virtual output of a DMU *j*; then the modified virtual input and output are determined by (9) and (10) respectively.

$$I'_j = \sum_i v'_{ij} x_{ij} \tag{9}$$

$$O'_j = \sum_i u'_{rj} y_{rj} \tag{10}$$

I'_j and O'_j are used to graphically represent any DMU *j* and the efficient frontier in a bi-dimensional plot. It is easy to see that the modified input weights, v'_{ij}, are in fact in compliance with the new restriction introduced, as $\sum_i v'_{ij} = 1$. The new normalization leads to the efficient frontier becoming a straight line from the origin that bisects the quadrant. This is because for a DMU to be efficient its modified virtual input must be equal to its modified virtual output, that is, for any DMU *j*, *j* is efficient if and only if $I'_j = O'_j$. Moreover, it is easy to see that the modified virtual input of any DMU *j*, I'_j, can be obtained from the standard virtual input, I_j, by using the expression $I'_j = \frac{I_j}{S_j}$. In an analogous way, $O'_j = \frac{O_j}{S_j}$. As both virtual input and virtual output for each DMU were divided by the same positive number, the efficiency value is not affected by the previous algebraic manipulations. In fact, the efficiency of a given DMU *o*, Eff_o, is obtained by dividing the virtual outputs by the virtual input. In (11) we show that this division is the same as the one obtained by dividing the modified virtual output by the modified virtual input.

$$Eff_o = \frac{O_o}{I_o} = \frac{\sum_r u_{ro} y_{ro}}{\sum_i v_{io} x_{io}} = \frac{\frac{\sum_r u_{ro} y_{ro}}{S_o}}{\frac{\sum_i v_{io} x_{io}}{S_o}} = \frac{\sum_r \frac{u_{ro}}{S_o} y_{ro}}{\sum_i \frac{v_{io}}{S_o} x_{io}} = \frac{\sum_r u'_{ro} y_{ro}}{\sum_i v'_{io} x_{io}} = \frac{O'_o}{I'_o} \qquad (11)$$

Let us note that the validation of the previous demonstration is related to the fact that $S_o \neq 0$.

So, the efficiency obtained by our modified model is the same obtained by the original fractional CCR-DEA input oriented model (1). As the fractional model and the linearized model (2) presents the same efficiency, the efficiency obtained by our modified model is the same obtained by the standard linearized model (2).

We summarize our proposed approach as a step by step procedure as done by Suzuki et al (2010), as follows.

Step 1. Run the standard input oriented CCR multiplier model shown in (2) for each DMU j ($\forall j$).

Step 2. Calculate the sum of all input weights (S_j, equation (6)), for each DMU j.

Step 3. Calculate the modified input and output weights, v'_{ij} and u'_{rj} , according to equations (7) and (8), respectively.

Step 4. Calculate the modified virtual input I'_j and output O'_j using equations (9) and (10) for each DMU j.

Step 5. Using the modified virtual input and output for each DMU j, plot all DMUs in a bi-dimensional graph in which I' is in the x-axis and O' is in the y-axis.

Step 6. Draw the 45º line representing the efficient frontier where efficient DMU present I'=O'.

At this point it is important to show that the proposed bi-dimensional representation of the DMUs and their distance from the efficient frontier could be used to geometrically determine the efficiency index as seen in Section 2. So, let P be an inefficient DMU, P_T the target or efficient projection of DMU P considering the input orientation; P_V is the projection of P on the y-axis and P_H is the projection of P on the x-axis; and P_{TH} is the projection of P_T on the x-axis. All these points are depicted in figure 4 and according to Farrell (1957) the efficiency for DMU P should be $Ef_P = \overline{P_V P_T} / \overline{P_V P}$.

By definition, the efficiency index of any DMU is the ratio between the virtual output and the virtual input. Thus, the efficiency of DMU P is defined as $Ef_P = \overline{OP_V} / \overline{OP_H}$.

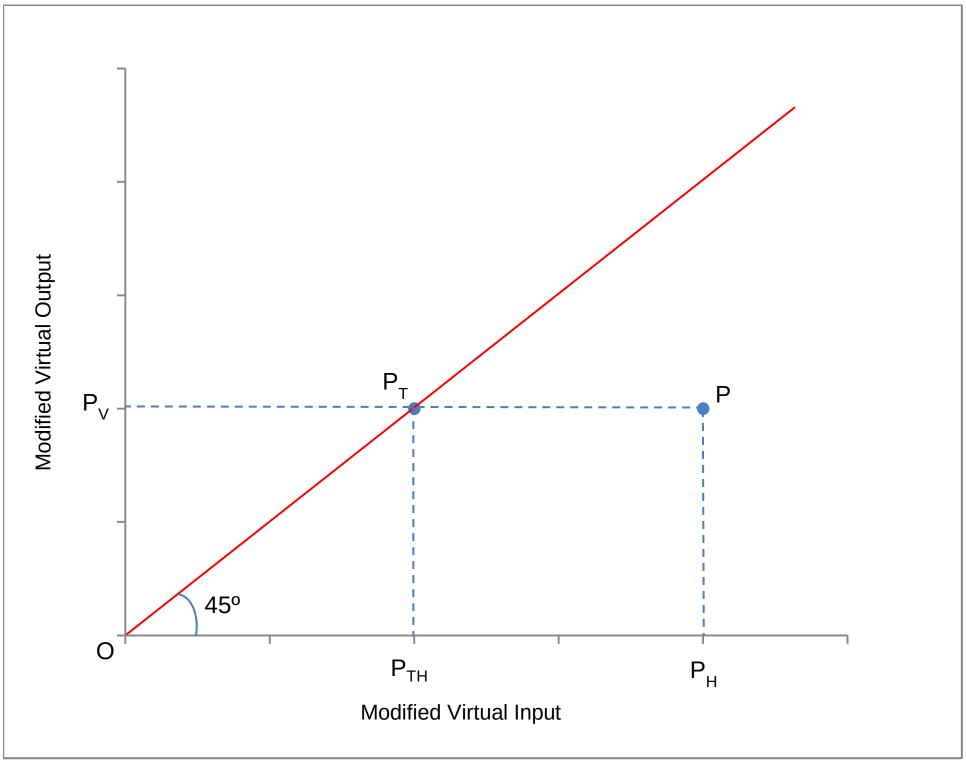

Figure 4. Geometrical representation of the efficiency.

From figure 4, it is easy to see that $\overline{OP_V} = \overline{P_H P} = \overline{P_{TH} P_T}$ and that $\overline{OP_H} = \overline{P_V P}$. Given that the efficient frontier is the bisectrix, by similarity of triangles $\overline{OP_{TH}} = \overline{P_V P_T}$ and $\overline{P_{TH} P_T} = \overline{OP_{TH}}$. Therefore, the efficiency of DMU P is given by (12).

$$Ef_P = \frac{\overline{OP_V}}{\overline{OP_H}} = \frac{\overline{P_H P}}{\overline{P_V P}} = \frac{\overline{P_{TH} P_T}}{\overline{P_V P}} = \frac{\overline{OP_{TH}}}{\overline{P_V P}} = \frac{\overline{P_V P_T}}{\overline{P_V P}} \qquad (12)$$

The efficiency of DMU P defined in (12) is the technical efficiency measure defined by Farrell (1957) as seen in Section 2.

4.2. The output oriented case

For the output oriented CCR model, the model is as shown in (13a) and the linearized model is in (13b). As can be seen the linearization constraint is related to the virtual output. Therefore, this virtual output is always equal to 1.

$$Min\ \frac{\sum_i v_i x_{io}}{\sum_r u_r y_{ro}}$$
$$s.t.$$
$$\frac{\sum_i v_i x_{ij}}{\sum_r u_r y_{rj}} \geq 1,\ \forall j$$
$$u_r \geq 0,\ v_i \geq 0,\ \forall r,i$$

(13a)

$$Min \sum_i v_i x_{io}$$
$$s.t.$$
$$\sum_r u_r y_{rj} = 1$$
$$\sum_r u_r y_{rj} - \sum_i v_i x_{ij} \leq 1,\ \forall j$$
$$u_r \geq 0,\ v_i \geq 0,\ \forall r,i$$

(13b)

In this case, we will use as S_j the total sum of the output weights of DMU *j*, as in (14).

$$S_j = \sum_i u_{ij} \qquad (14)$$

And in the step by step procedure for the output oriented CCR model we will replace the step 2 by the following step 2 for output oriented models.

Step 2 (for output oriented models). Determine the sum of all output weights (S_j, equation (14)), for each DMU *j*.

5. Numerical Example

The proposed bi-dimensional representation is exemplified with data from six DMUs, two inputs and two outputs, the values of which are presented in table 1. Using the procedure shown in the previous section, step 1 is to run the standard input oriented CCR model, we have used SIAD software (Angulo-Meza et al., 2005) or any available DEA software or linear programming solver. The results of this step are also depicted in table 1.

Table 1. Data, efficiency index and weights using the standard input oriented CCR model.

DMU	*Variables*				*Efficiency Index*	*Weights*			
	x_1	x_2	y_1	y_2		v_1	v_2	u_1	u_2
A	1	4	2	6	1.0000	0.2381	0.1905	0.0000	0.1667
B	3	4	9	5	1.0000	0.1556	0.1333	0.1111	0.0000

C	4	2	3	8	1.0000	0.1786	0.1429	0.0000	0.1250
D	6	8	5	3	0.2862	0.1014	0.0489	0.0344	0.0380
E	2	1	4	3	1.0000	0.3500	0.3000	0.2500	0.0000
F	1	9	1	1	0.3636	1.0000	0.0000	0.2955	0.0682

We must point out that using different DEA packages or linear programming solvers we will the same efficiency indexes but for the efficient DMU we could obtain different multipliers (multiple optimal solutions). We will address this issue in section 8.

With these results, step 2 (for input oriented models) is implemented, calculating S_j, second column of table 2. In step 3, using results in the previous step, we calculate the modified inputs (v'_i) and output (u'_r) weights, which are depicted from the third to sixth columns of the same table. In step 4 we calculate the modified virtual inputs (I') and outputs (O'), which are in the seventh and eighth columns of table 2. In this same table the efficiency (O'/I') index is shown.

Table 2. Normalized weights, modified virtual input and output and efficiency index for the input oriented CCR model

DMU	S_j	*Modified weights*				*Modified Virtual input and output*		*Efficiency Index*
		v'_1	v'_2	u'_1	u'_2	I'	O'	
A	0.4286	0.5556	0.4444	0.0000	0.3889	2.3333	2.3333	1.0000
B	0.2889	0.5385	0.4615	0,3846	0.0000	3.4615	3.4615	1.0000
C	0.3214	0.5556	0.4444	0.0000	0.3889	3.1111	3.1111	1.0000
D	0.1504	0.6747	0.3253	0.2862	0.2530	6.6506	1.9036	0.2862
E	0.6500	0.5385	0.4615	0,3846	0.0000	1.5385	1.5385	1.0000
F	1.0000	1.0000	0.0000	0.2955	0.0682	1.0000	0.3636	0.3636

With steps 5 and 6 we obtain figure 5, which depicts the efficient frontier and the DMUs' location, using their modified virtual input and virtual output given in Table 2. All DMUs on the bisectrix are efficient (A, B, C and E) and those below this frontier are inefficient (F and D).

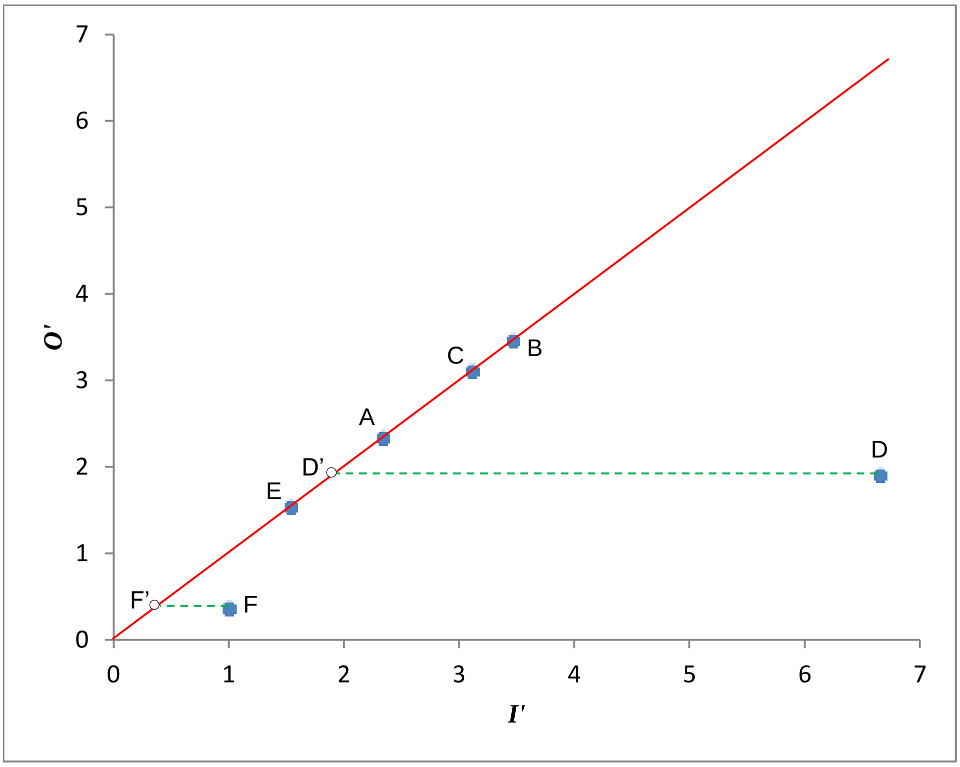

Figure 5. Proposed bi-dimensional representation of DMUs, input oriented CCR model.

As demonstrated in the previous section, the efficiency calculated using I' and O', that is O'/I', is exactly the same as calculating I'^{*}/I'. Indeed, for the ineficient DMUs D and F, the targets in the input orientation are $I_D'^{*}$ = 1.9036 and $I_F'^{*}$ = 0.3636, in a horizontal move to the frontier, which equals O_D' and O_F' .

As mentioned in the previous section, for the output orientation, only step 2 is modified. Therefore, we first run the standard CCR output oriented model (13b), step 1, which data and results are depicted in table 3.

Table 3. Data, efficiency index and weights using the standard output oriented CCR model.

DMU	*Variables*				*Objective Function Value*	*Efficiency Index*	*Weights*			
	x_1	x_2	y_1	y_2			v_1	v_2	u_1	u_2
A	1	4	2	6	1.0000	1.0000	0.2381	0.1905	0.0000	0.1667
B	3	4	9	5	1.0000	1.0000	0.1556	0.1333	0.1111	0.0000
C	4	2	3	8	1.0000	1.0000	0.1786	0.1429	0.0000	0.1250
D	6	8	5	3	3.4941	0.2862	0.3544	0.1709	0.1203	0.1329
E	2	1	4	3	1.0000	1.0000	0.4029	0.1942	0.1367	0.1511
F	1	9	1	1	2.7503	0.3636	2.7500	0.0000	0.8125	0.1875

The efficiency index is the same as in the input orientation provided that the efficiency index in the output oriented model is the inverse of the optimal objective function value (Coelli et al., 1998, pages 55-56; Cooper et al., 2007, page 58). Although, the efficiency indexes are the same for both orientation, inputs and outputs weights are different, as well as the modified weights.

In step 2 for output oriented models, $S_{j,}$ is ascertained as in (14), i.e., the sum of the output weights for each DMU j. These results and also the results for steps 3, 4, 5 and 6 are depicted in Table 4.

Table 4. Normalized weights and modified virtual input and output for the output oriented CCR model

DMU	S_j	*Modified weights*				*Modified Virtual input and output*		*Efficiency Index*
		v'_1	v'_2	u'_1	u'_2	I'	O'	
A	0.1667	1.4286	1.1429	0.0000	1.0000	6.0000	6.0000	1.0000
B	0.1111	1.4000	1.2000	1.0000	0.0000	9.0000	9.0000	1.0000
C	0.1250	1.4286	1.1429	0.0000	1.0000	8.0000	8.0000	1.0000
D	0.2532	1.4000	0.6750	0.4750	0.5250	13.8000	3.9500	0.2862
E	0.2878	1.4000	0.6750	0.4750	0.5250	3.4750	3.4750	1.0000
F	1.0000	2.7500	0.0000	0.8125	0.1875	2.7500	1.0000	0.3636

Figure 6 shows the modified virtual input and virtual output for the output oriented CCR model and the efficient frontier. As in the input oriented case, DMUs A, B, C and E are efficient and D and F are inefficient.

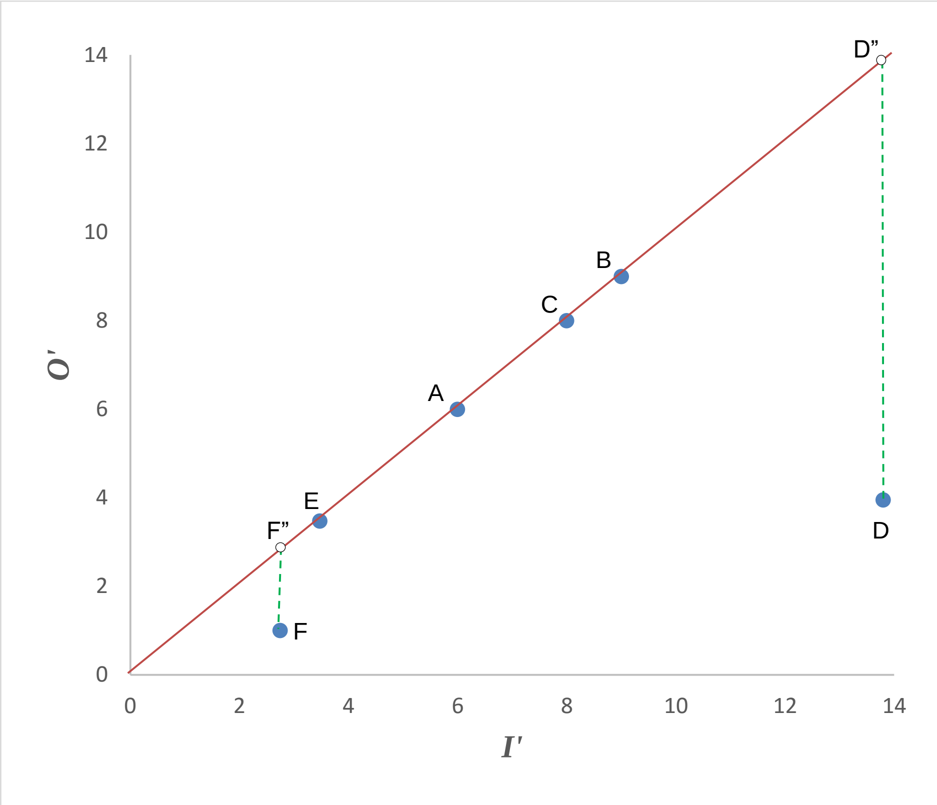

Figure 6. Proposed bi-dimensional representation of DMUs for the output oriented CCR model.

As in the input oriented case, Figure 6 clearly shows that the geometrical distance from the efficient frontier is the DMUs' efficiency. Now, the targets in the output orientation are $O_D'^* = 13.8$ and $O_F'^* = 2.75$, in a vertical move to the frontier. These targets are depicted in Figure 6: D" and F".

6. The proposed graphical representation for the BCC model

As the proposed approach uses a modified virtual input and virtual output to produce a bi-dimensional plot of the DMUs and the efficiency values, it can also be applied to the BCC model presented in (3). In fact, the additional restriction is the same (the sum of the input weights equals 1) and the resulting input oriented BCC model is shown in (15).

$$
\begin{aligned}
& Max \ \frac{\sum_r u_r y_{ro} + u_*}{\sum_i v_i x_{io}} \\
& s.t. \\
& \sum_i v_i = 1 \\
& \frac{\sum_r u_r y_{rj} + u_*}{\sum_i v_i x_{ij}} \leq 1, \ \forall j \\
& u_r \geq 0, \ v_i \geq 0, \ \forall r, i \\
& u_* \in \Re
\end{aligned}
\qquad (15)
$$

As in the case of the CCR model, it is not necessary to use this fractional formulation. Instead all the weights of the standard input oriented BCC model, including u_*, are divided by the total sum of the input weights. So, for the scale factor, u_*, the normalization is as in (16)

$$u'_{*j} = \frac{u_{*j}}{S_j} \qquad (16)$$

The new virtual input is calculated by (9) and the virtual output is obtained using equation (17), similar to (10).

$$O'_j = \sum_i u'_{rj} y_{rj} + u'_{*j} \qquad (17)$$

Therefore, the proposed step by step procedure for the BCC model is as follows.

Step 1. Run the standard input oriented BCC multiplier model shown in (4) for each DMU j ($\forall j$).

Step 2. Calculate the sum of all input weights (S_j, equation (6)), for each DMU j.

Step 3. Calculate the modified input and output weights, v'_{ij}, u'_{rj} and u'_{*j}, according to equations (7), (8) and (16), respectively.

Step 4. Calculate the modified virtual input I'_j and output O'_j using equations (9) and (17) for each DMU j.

Step 5. Using the modified virtual input and output for each DMU j, plot all DMUs in a bi-dimensional graph in which I' is in the x-axis and O' is in the y-axis.

Step 6. Draw the 45º line representing the efficient frontier, where efficient DMUs present I'=O'.

Using the data from Table 1 and the SIAD software, the results for the input oriented BCC model were obtained and depicted in Table 5. In this table, we can see that only DMU D is inefficient.

Table 5. Data, efficiency index and weights for the input oriented BCC model.

DMU	*Variables*				*Efficiency Index*	*Weights*				
	I_1	I_2	O_1	O_2		v_1	v_2	u_1	u_2	u_*
A	1	4	2	6	1.0000	1.0000	0.0000	0.0000	0.1667	0.0000
B	3	4	9	5	1.0000	0.3333	0.0000	0.0952	0.0000	0.1429
C	4	2	3	8	1.0000	0.1853	0.1295	0.0692	0.1138	-0.1183
D	6	8	5	3	0.3400	0.1400	0.0200	0.0400	0.0000	0.1400
E	2	1	4	3	1.0000	0.4667	0.0667	0.1333	0.0000	0.4667
F	1	9	1	1	1.0000	1.0000	0.0000	0.0000	0.0000	1.0000

We shall note that DMU F presents the weights of both outputs nil valued and the virtual output has the scale factor, u_*, as the only non-zero item. This is because DMU F is weakly efficient.

Using the proposed approach, the results shown in Table 6 are obtained: *S*, the modified weights and the modified virtual input and output.

Table 6. Normalized weights and modified virtual input and output for the input oriented BCC model

DMU	*S*	*Modified weights*					*Modified Virtual input and output*	
		v_1'	v_2'	u_1'	u_2'	u_*'	I'	O'
A	1.0000	1.0000	0.0000	0.0000	0.1667	0.0000	1.0000	1.0000
B	0.3333	1.0000	0.0000	0.2857	0.0000	0.4286	3.0000	3.0000
C	0.3147	0.5887	0.4113	0.2199	0.3617	-0.3759	3.1773	3.1773
D	0.1600	0.8750	0.1250	0.2500	0.0000	0.8750	6.2500	2.1250
E	0.5333	0.8750	0.1250	0.2500	0.0000	0.8750	1.8750	1.8750
F	1.0000	1.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000

For the data in Table 6, the efficient frontier and the DMUs' location are shown in Figure 7, clearly highlighting that DMUs A, B, C, E and F are efficient and only DMU D is inefficient.

It is worth noticing in Figure 7 that DMUs A and F coincide at the same point on the efficient frontier. The fact that different DMUs (efficient, as in this example, or not) are represented by the same point may be seen as a disadvantage of the proposed graphical representation. Nevertheless, the relevance of the graph is in the clear visualisation of the position of the DMUs in relation to the efficient frontier. Note that a similar disadvantage also happens when representing the efficient frontier of the CCR model with three variables (see Section 2) for two different DMUs with the same ratios O1/I and O2/I.

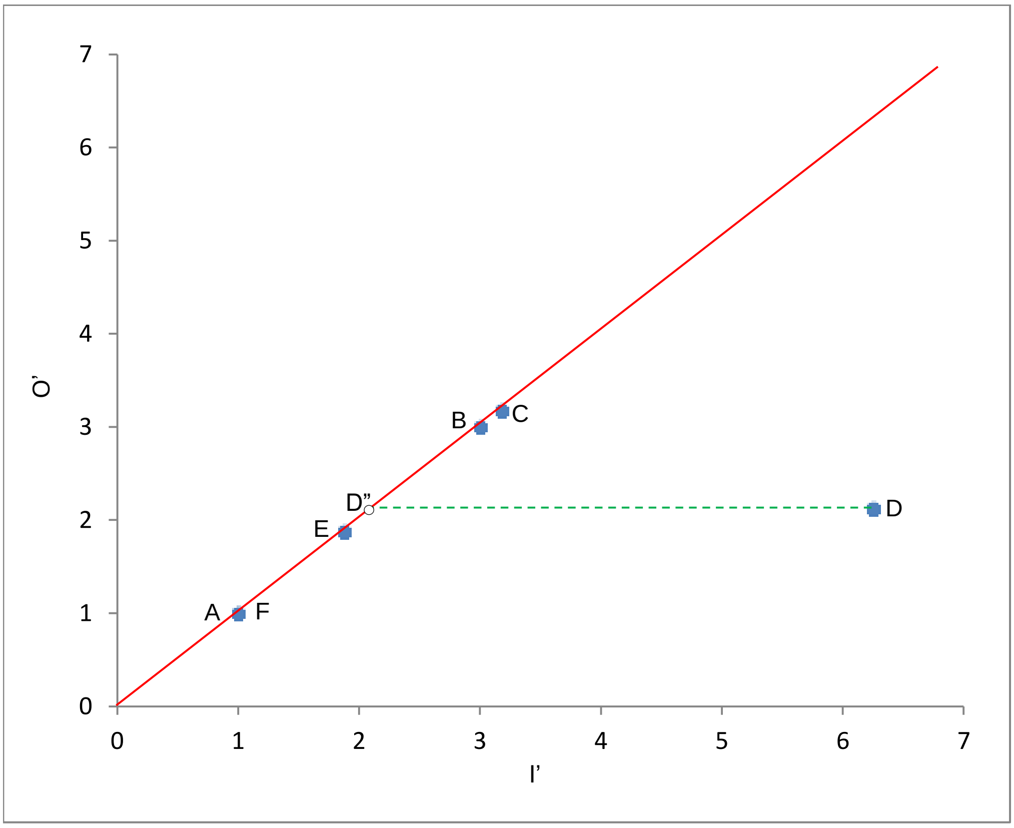

Figure 7. Proposed bi-dimensional representation of DMUs for the input oriented BCC frontier

The target for the modified virtual input of DMU D is obtained through the horizontal projection on the frontier, that is, the virtual input reduction $I_D'^*$ = 2.1250. This target is depicted in Figure 7 as D'.

It is relevant to present the proposed normalization for the output oriented BCC model (18), in which the term v_* appears in the virtual input).

$$\begin{aligned}
& Min \ \frac{\sum_i v_i x_{io} + v_*}{\sum_r u_r y_{ro}} \\
& s.t. \\
& \frac{\sum_i v_i x_{ij} + v_*}{\sum_r u_r y_{rj}} \geq 1, \ \forall j \\
& u_r \geq 0, \ v_i \geq 0, \ \forall r, i \\
& v_* \in \Re
\end{aligned} \qquad (18)$$

As in the case of the output oriented CCR model, we will use as S_j the total sum of the output weights of DMU j, as in (14).

$$S_j = \sum_i u_{ij} \qquad (19)$$

Thus, all weights will be divided by S_j, including the scale factor v_* as in (20).

$$v'_{*j} = \frac{v_{*j}}{S_j} \qquad (20)$$

Therefore, the modified virtual input is calculated using equation (21) similar to equation (17) and the modified virtual output is calculated using (10).

$$I'_j = \sum_i v'_{ij} x_{ij} + v'_{*j} \qquad (21)$$

And in the step by step procedure for the output oriented BCC model we will replace the step 2 and 3 by the following steps 2 and 3 for output oriented models (observe that Step 2 is similar to Step 2 for the output oriented CCR model).

Step 2 (for output oriented models). Calculate the sum of all input weights (S_j, equation (19)), for each DMU j.

Step 3. Calculate the modified input and output weights, v'_{ij}, u'_{rj} and u'_{*j}, according to equations (7), (8) and (20), respectively.

The data in Table 7 were used to illustrate the proposed approach for the output oriented BCC model. As before, the values of the efficiency index and of the weights were calculated using the SIAD software. Now, D and F are the inefficient DMUs.

Table 7. Data, efficiency index and weights for the output oriented BCC model.

DMU	*Variables*				*Efficiency Index*	*Weights*				
	I_1	I_2	O_1	O_2		v_1	v_2	v_*	u_1	u_2
A	1	4	2	6	1.0000	1.0000	0.0000	0.0000	0.0000	0.1667
B	3	4	9	5	1.0000	0.0949	0.0000	0.7152	0.0443	0.1203
C	4	2	3	8	1.0000	0.0867	0.0000	0.6532	0.0405	0.1098
D	6	8	5	3	0.5789	0.0000	0.0000	1.7273	0.0909	0.1818
E	2	1	4	3	1.0000	0.0000	0.3553	0.6447	0.1711	0.1053
F	1	9	1	1	0.5000	3.5000	0.0000	-1.5000	1.0000	0.0000

The modified weights are determined using equations (7), (8) and (20) and the modified virtual input and virtual output are determined using equations (21) and (10). The modified weights and the modified virtual input and virtual output are depicted in Table 8. Also, the modified virtual input and virtual output are shown in Figure 8.

Table 8. Normalized weights and new virtual input and output using the total sum of the outputs weights.

DMU	*S*	*Modified weights*					*Modified Virtual input and output*	
		v'_1	v'_2	v'_*	u'_1	u'_2	I'	O'
A	0.1667	6.0000	0.0000	0.0000	0.0000	1.0000	6.0000	6.0000
B	0.1646	0.5765	0.0000	4.3451	0.2691	0.7309	6.0765	6.0765
C	0.1503	0.5768	0.0000	4.3460	0.2695	0.7305	6.6527	6.6527
D	0.2727	0.0000	0.0000	6.3341	0.3333	0.6667	6.3341	3.6667
E	0.2764	0.0000	1.2855	2.3325	0.6190	0.3810	3.6190	3.6190
F	1.0000	3.5000	0.0000	-1.5000	1.0000	0.0000	2.0000	1.0000

We can also see that the efficiency index of every DMU is the modified virtual output divided by the modified virtual input, i.e. O'/I', and that $O' = I'$ for the efficient DMUs.

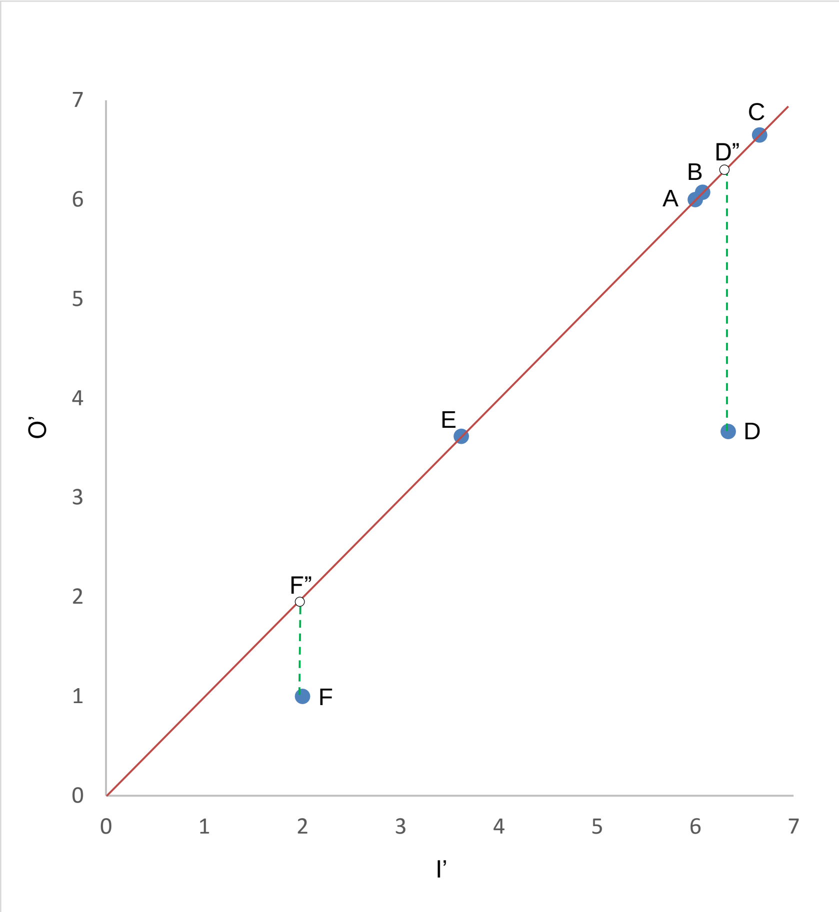

Figure 8. Proposed bi-dimensional representation of DMUs for the output oriented BCC frontier

7. Determining the efficiency of DMUs from a bi-dimensional graphical representation

In order to illustrate the concepts presented hereabove with real data, we will use the data from Adler and Raveh (2008). In such paper is used a data frequently analysed in DEA, where 35 Chinese cities were analysed using as inputs labor force, working funds and investment and as outputs gross industrial output, profit and taxes, and retail sales. Using the CCR model and our proposed approach for bi-dimensional graphical representation, we have obtained figure 9.

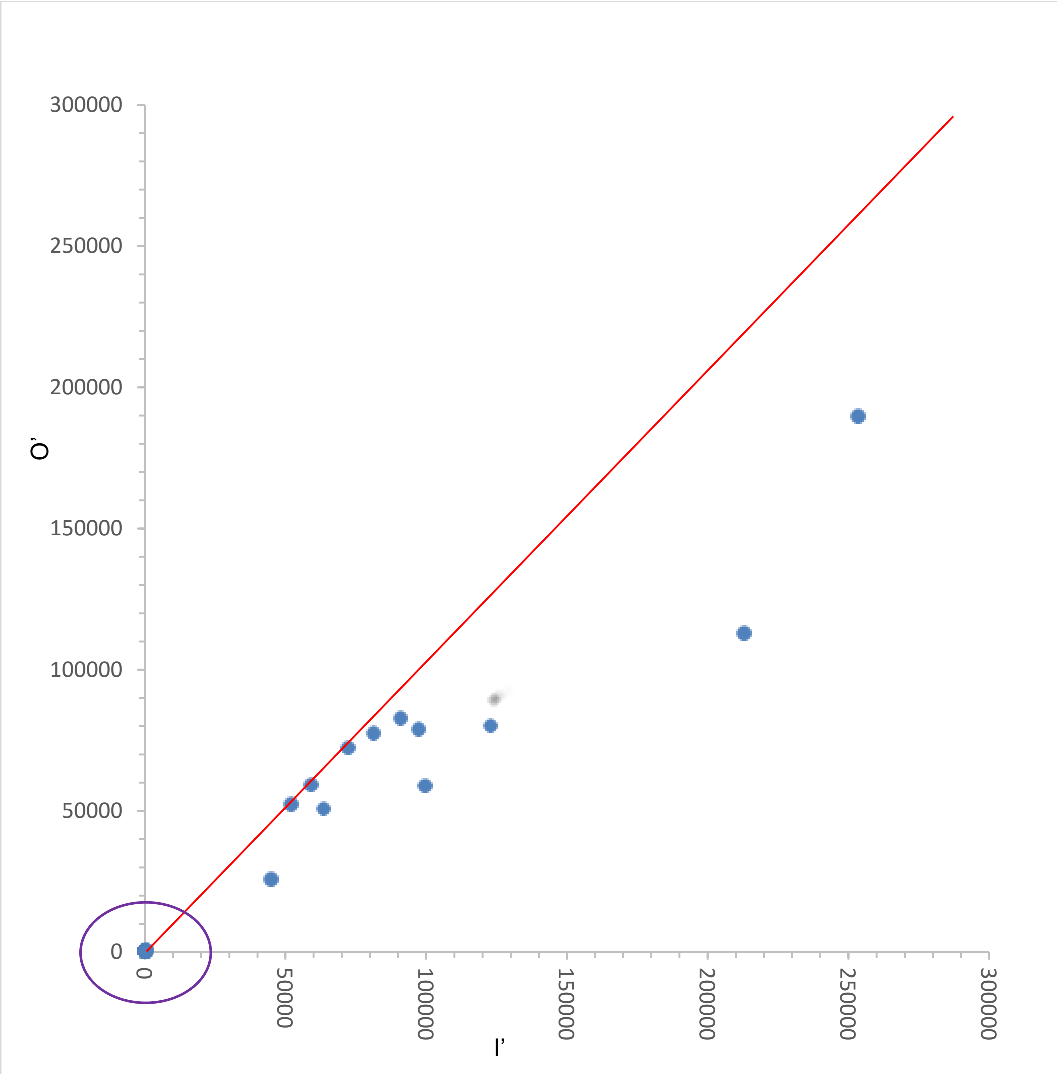

Figure 9. Bi-dimensional representation of data in Adler and Raveh (2008)

In this figure, even though it is not possible to visually identify there is a large number of DMU near the (0,0). To better visualize these DMUs, the region circled in figure 9 is zoomed in figure 10.

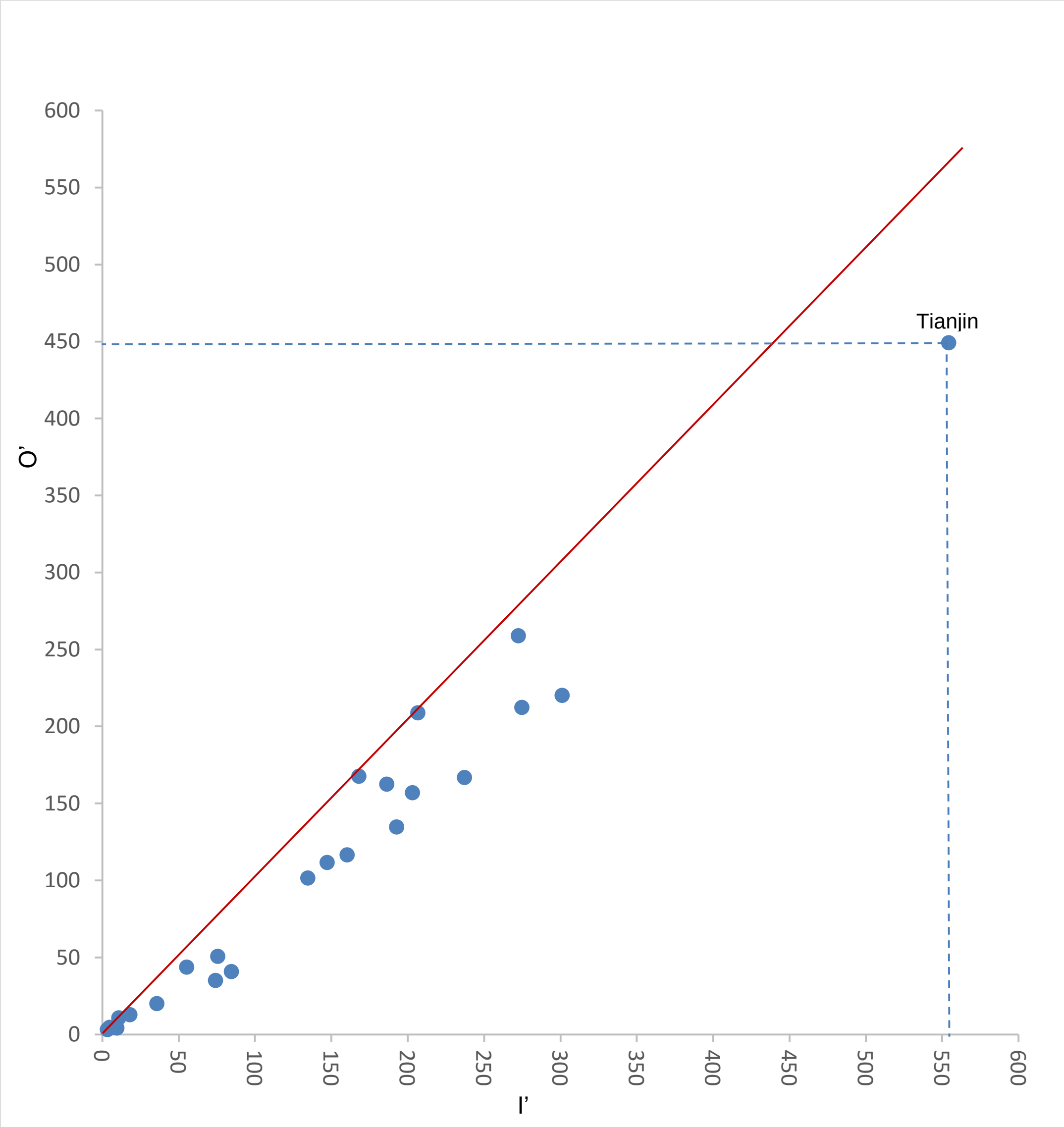

Figure 10. Bi-dimensional representation for “small” DMUs.

A manager or decision maker presented with both figures can have an overall perception of the DMUs distribution and their efficiency.

Although not being the main objective of the graphical representation, a manager with no access to the efficiency indexes is able to determine a DMU efficiency dividing the virtual output (y-axis) by the virtual input (x-axis). For instance, for DMU Tianjin, the efficiency is approximately 450/550=0,8181 which is very close to the actual value 0,8145.

At this point we shall note that DMUs in figure 10 were called “small” DMUs. This brings the issue of a DMU “size”. Implicitly, we have used the intuitive idea that a “small” DMU is near the (0,0) point and conversely a “large” DMU is far from the same point. However, there are two issues to formalize such concept. The first one is that the virtual input and virtual output, as well as the modified virtual input and output, depend obviously on the multiplier assigned to each DMU. Such multipliers are determined in order to maximize the efficiency score of the DMU, and its use in size measure is quite debateable. The second problem arises only for efficient DMU. As it is to be seen in the next section, for such DMU the graphical representation is a line not a fixed point. As a consequence, the "size" of an efficient DMU is not uniquely determined. These last issue can be solved using the smooth DEA (Naciff et al, 2009; Brandon and Soares de Mello, 2016) approach but only for BCC models. A more general approach would be using a common set of weights approach in the lines of (Carvalho, 2016). Both approaches would change the DMUs efficiency scores and consequently their graphical representation.

8. Analysis of the multiple optimal weights

As it is common knowledge for the efficient DMUs there are multiple optimal solutions for the weights as pointed out by Sexton et al (1986), Doyle and Green (1994), Rosen et al (1998) and Soares de Mello et al (2002) among others. Even though the location of efficient DMUs may change along the 45º line, the representation of inefficient DMUs remains always in the same location on the graph. Therefore, the proposed graphical representation of this paper remains useful to visualize the frontier and the relative position of the inefficient DMUs.

For the efficient DMUs it would be interesting to determine all possible location in the efficient frontier. For that end, we need to determine the lowest modified virtual input I' and the highest modified virtual input I', being that as we are dealing only with efficient DMUs I'=O'.

Then, for the input oriented CCR model and for the efficient DMU *o* we want to

$$\text{Min (or Max) } I'_o = \text{Min (or Max) } \sum_i v'_{io} x_{io} \qquad (22)$$

This optimizing problem is constrained with the usual CCR input oriented constraints using the modified weights instead of the original weights as in (4). Furthermore, there is also the constraint imposing DMU *o* to be efficient and the usual normalization constraint is replaced by the constraint the sum of the modified weights equals 1.

$$\begin{aligned}
& Min\left(or\ Max\right)\sum_i v'_{io}x_{io} \\
& st \\
& \sum_i v'_{io} = 1 \\
& \sum_r u'_{ro}\, y_{ro} - \sum_i v'_{io}\, x_{io} = 0 \\
& \sum_r u'_{rj}\, y_{rj} - \sum_i v'_{ij}\, x_{ij} \le 0,\ \forall j \\
& u'_{rj}, v'_{ij} \ge 0
\end{aligned} \qquad (23)$$

For instance, using data in Table 1, for the efficient DMU A, model (23) becomes model (24).

$$\begin{aligned}
& Min\left(or\ Max\right) v'_{1A} + 4v'_{2A} \\
& st \\
& v'_{1A} + v'_{2A} = 1 \\
& 2u'_{1A} + 6u'_{2A} - v'_{1A} - 4v'_{2A} = 0 \\
& 2u'_{1A} + 6u'_{2A} - v'_{1A} - 4v'_{2A} \le 0 \\
& 9u'_{1A} + 5u'_{2A} - 3v'_{1A} - 4v'_{2A} \le 0 \\
& 3u'_{1A} + 8u'_{2A} - 4v'_{1A} - 2v'_{2A} \le 0 \\
& 5u'_{1A} + 3u'_{2A} - 6v'_{1A} - 8v'_{2A} \le 0 \\
& 4u'_{1A} + 3u'_{2A} - 2v'_{1A} - v'_{2A} \le 0 \\
& u'_{1A} + u'_{2A} - v'_{1A} - 9v'_{2A} \le 0 \\
& u'_{rj}, v'_{ij} \ge 0
\end{aligned} \qquad (24)$$

Where the third constraint, regarding efficient DMU A, is obviously redundant. The results for this DMU are depicted in Table 9.

Table 9. Min and max inputs weights for DMU A

Model	*Modified weights*		*Modified Virtual input, I'*
	v'_{1A}	v'_{2A}	
Min	1,0000	0,0000	1,0000
Max	0,5556	0,4444	2,3333

For all the efficient, the min and max values for I’ are shown in Table (10).

Table (10). Min and max values for the modified virtual inputs of efficient DMU

DMU	*Modified Virtual input, I'*	
	Min	Max
A	1,0000	2,3333
B	3,0000	3,4615
C	2,0000	3,1262
E	1,0000	1,6747

With these results and the results in table 2 for the inefficient DMUs, the graphical representation taking into account the multiple optimal weights for efficient DMUs are shown in Figure 11. We shall note that the inefficient DMUs are still represented by a single point in the same location and that the generic efficient DMU o is represented by the segment $\overline{oO}$. As an example we have highlighted the segment representing DMU A.

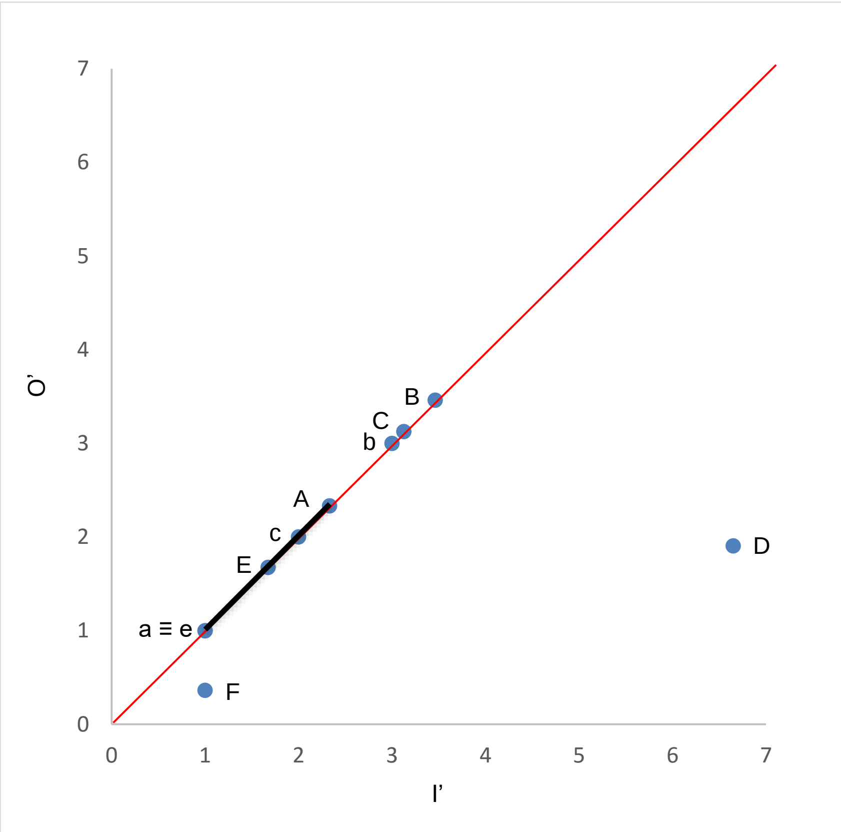

Figure 11. Bi-dimensional representation and segment for DMU A.

At this point we shall point out that if necessary, it is possible to obtain the min and max I' using a linear programme dealing with the original weights instead of the modified weights. In fact, for any DMU j, from (6), (7), (8) and (9).

$$I'_j = \sum_i v'_{ij} x_{ij} = \sum_i \left(\frac{v_{ij}}{S_j} \right) x_{ij} = \sum_i \left(\frac{v_{ij}}{\sum_i v_{ij}} \right) x_{ij} = \frac{\sum_i v_{ij} x_{ij}}{\sum_i v_{ij}} \qquad (25)$$

As we are dealing with the original weights instead of the modified weights, the normalization is the usual as in standard input oriented CCR model (2). Therefore

$$I'_j = \frac{\sum_i v_{ij} x_{ij}}{\sum_i v_{ij}} = \frac{1}{\sum_i v_{ij}} \qquad (26)$$

This means that maximizing (minimizing) the modified virtual input is equivalent to minimizing (maximizing) the sum of the original input weights.

The complete model with standard CCR constraints and there is also the constraint imposing DMU *o* to be efficient is presented in (27).

$$\begin{aligned}
& Max(or\ Min) \sum_i v_{io} \\
& st \\
& \sum_i v_{io} x_{io} = 1 \\
& \sum_r u_{ro} y_{ro} - \sum_i v_{io} x_{io} = 0 \\
& \sum_r u_{rj} y_{rj} - \sum_i v_{ij} x_{ij} \le 0,\ \forall j \\
& u_{rj}, v_{ij} \ge 0
\end{aligned} \qquad (27)$$

Exemplifying for DMU A, we obtain model (28),

$$\begin{aligned}
& Max(or\ Min) v_{1A} + v_{2A} \\
& st \\
& 1v_{1A} + 4v_{2A} = 1 \\
& 2u_{1A} + 6u_{2A} - v_{1A} - 4v_{2A} = 0 \\
& 2u_{1A} + 6u_{2A} - v_{1A} - 4v_{2A} \le 0 \\
& 9u_{1A} + 5u_{2A} - 3v_{1A} - 4v_{2A} \le 0 \\
& 3u_{1A} + 8u_{2A} - 4v_{1A} - 2v_{2A} \le 0 \\
& 5u_{1A} + 3u_{2A} - 6v_{1A} - 8v_{2A} \le 0 \\
& 4u_{1A} + 3u_{2A} - 2v_{1A} - v_{2A} \le 0 \\
& u_{1A} + u_{2A} - v_{1A} - 9v_{2A} \le 0 \\
& u_{rj}, v_{ij} \ge 0
\end{aligned} \qquad (28)$$

Moreover, for that DMU the results are shown in Table 11.

Table 11. Inputs weights and modified virtual input for DMU A

Model	*Weights*		$v_{1A} + v_{2A}$	$\frac{1}{v_{1A} + v_{2A}} = I'$
	v_{1A}	v_{2A}		
Max	1,000000	0,000000	1,000000	1,0000
Min	0,238095	0,190476	0,4285714	2,3333

Clearly, the same model can be used for the other efficient DMUs and easily adapted for output orientation and variables returns to scale.

9. Conclusions

In this paper, a new approach to graphically represent the efficient frontier and the DMUs' location was presented. Initially, a new restriction to normalize the DEA weights was introduced, which can be used in both the CCR and the BCC models. The new restriction generates a fractional formulation; however, there is no need to run this new formulation, because instead all the resulting weights obtained by the standard CCR or BCC models are divided by the total sum of the input weights. The modified weights are used to calculate a new (or modified) virtual input and virtual output, which are then used in a bi-dimensional plot to determine the efficient frontier and the position of the DMUs in relation to this frontier. As in the standard CCR and BCC models with two variables, the DMUs that are on the frontier are efficient and those below the frontier are inefficient. In this graphical representation, the efficiency is calculated geometrically as in the CCR and BCC models.

This new approach has advantages over other approaches: the efficient frontier is clearly defined as is the DMUs' location in relation to this frontier; there is no need to run a new model, just a simple division of the total sum of the input weights by the standard DEA weights; it provides the same efficiency index calculated geometrically as in the CCR and BCC models, whereby the distance of the DMU from the frontier provides an idea of how inefficient this DMU is; non geometrically, the efficiency index of a DMU is the modified virtual output divided by the modified virtual input, analogous to the efficiency index calculated using the CCR and BCC models. On the other hand, due to the characteristic of the graphical representation, it is not possible to graphically determine the reference set of an inefficient DMU.

It should be noted that no matter which model is used; the frontier does not move. In contrast to the standard DEA graphical representations, the DMUs are represented by different points depending on the model orientation and the graphical representation of the frontier always remains the same.

Research is being carried out to extend this approach to other DEA models, such as DEA models with weight restrictions, and the non-radial projections as done by Soares de Mello et al (2012), Gomes Jr et al (2013).

Another line for future studies is a definition of DMU’s size consistent with the proposed bi-dimensional graphical representation.

Acknowledgements

The authors want to thank the valuable comments of participants present in the IFORS 2014 Conference and the Portuguese and Brazilian science institutions, FCT, CAPES and CNPq, for the support given to the research presented in this article that was partially undertaken during post-doctoral of the second and third authors at CEG-IST in Lisbon. We also would like to thank the referees for their valuable comments and suggestions

References

ADLER, N. & RAVEH, A. 2008. Presenting DEA graphically. *Omega,* 36**,** 715.

AKÇAY, A. E., ERTEK, G. & BÜYÜKÖZKAN, G. 2012. Analyzing the solutions of DEA through information visualization and data mining techniques: SmartDEA framework. *Expert Systems with Applications,* 39**,** 7763-7775.

ANGULO-MEZA, L., BIONDI NETO, L., SOARES DE MELLO, J. C. C. B., GOMES, E. G. & COELHO, P. H. G. Free software for Decision Analysis a software package for Data Envelopment models. ICEIS 2005 - Proceedings of the 7th International Conference on Enterprise Information Systems, 2005 Miami. 207-212.

APPA, G., BANA E COSTA, C. A., CHAGAS, M. P., FERREIRA, F. C. & SOARES, J. O. 2010. DEA in X-factor evaluation for the Brazilian Electricity Distribution Industry. *Working Paper LSEOR 12-121.* London School of Economics, U.K.

BANA E COSTA, C. A., SOARES DE MELLO, J. C. C. B. & ANGULO MEZA, L. A New Approach to the Bi-Dimensional Representation of the DEA Efficient Frontier with Multiple Inputs and Outputs 20th Conference of the International Federation of Operational Research Societies - IFORS, 2014 Barcelona.

BANKER, R. D., CHARNES, A. & COOPER, W. W. 1984. Some models for estimating technical scale inefficiencies in data envelopment analysis. *Management Science,* 30**,** 1078-1092.

BELTON, V. & VICKERS, S. P. 1992. V.I.S.A. II - Visual Interactive Sensitivity Analysis for Multiple Criteria Decision Making. *User Manual.*

BELTON, V. & VICKERS, S. P. 1993. Demystifying DEA – A Visual Interactive Approach based on Multiple Criteria Analysis. *Journal of the Operational Research Society,* 44**,** 883-896.

BOUGNOL, M. L., DULÁ, J. H., ESTELLITA LINS, M. P. & MOREIRA DA SILVA, A. C. 2010. Enhancing standard performance practices with DEA. *Omega,* 38**,** 33-45.

BRANDÃO, L.C. ; SOARES DE MELLO, J. C.C.B. 2016. Improvements to Smooth Data Envelopment Analysis. RAIRO Operational Research. DOI: 10.1051/ro/2016012.

CARVALHO, N. P. 2016. Relative size and csw-efficiency: two new concepts based on DEA to guide management of capital. PhD Thesis. Federal Fluminense University.

CHARNES, A., COOPER, W. W. & RHODES, E. 1978. Measuring the efficiency of decision-making units. *European Journal of Operational Research,* 2**,** 429-444.

COELLI, T., RAO D.S.P. & BATTESE, G. E. 1998. *An introduction to efficiency and productivity analysis*, Kluwer Academic Publishers.

COOPER, W. W., SEIFORD, L. M. & TONE, K. 2007. *Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software*.

COOPER, W. W., SEIFORD, L. M. & ZHU, J. 2004. *Handbook on Data Envelopment Analysis,* New York, Boston, Dordrecht, London, Moscow, Kluwer Academic Publishers.

DEBREU, G. 1951. The Coefficient of Resource Utilisation. *Econometrica,* 19**,** 273-292.

DESAI, A. & WALTERS, L. C. 1991. Graphical Presentations of Data Envelopment Analyses: Management Implications from Parallel Axes Representations. *Decision Sciences,* 22**,** 335-353.

DOYLE, J. & GREEN, R. H. 1994. Efficiency and cross-efficiency in DEA derivations, meanings and uses. *Journal of the Operational Research Society,* 45**,** 567-578.

EL-MAHGARY, S. & LAHDELMA, R. 1995. Data envelopment analysis: Visualizing the results. *European Journal of Operational Research,* 83**,** 700-710.

FARRELL, M. J. 1957. The Measurement of Productive Efficiency. *Journal of Royal Statistical Society Series A,* 120**,** 253-281.

GOMES JUNIOR, S.F. ; SOARES DE MELLO, J.C.C.B. ; ANGULO-MEZA, L. 2013. DEA non-radial efficiency based on vector properties. International Transactions in Operational Research, 20, 341-364.

HACKMAN, S. T., PASSY, U. & PLATZMAN, L. K. 1994. Explicit Representation of the Two-Dimensional Section of a Production Possibility Set. *Journal of Productivity Analysis,* 5**,** 161-70.

HUANG, H. & LIAO, W. 2012. A Co-Plot-based efficiency measurement to commercial banks. *Journal of Software,* 7**,** 2247-2251.

MAITAL, S. & VANINSKY, A. 1999. Data envelopment analysis with a single DMU: A graphic projected-gradient approach. *European Journal of Operational Research,* 115**,** 518-528.

NACIF, F.B. ; SOARES DE MELLO, J.C.C.B. ; ANGULO-MEZA, L. 2009. Choosing weights in optimal solutions for DEA-BCC models by means of a n-dimensional smooth frontier. Pesquisa Operacional, 29 (3), 623-642.

OZCAN, Y. A., LINS, M. E., LOBO, M. S. C., DA SILVA, A. C. M., FISZMAN, R. & PEREIRA, B. B. 2010. Evaluating the performance of Brazilian university hospitals. *Annals of Operations Research,* 178**,** 247-261.

RAVEH, A. 2000. Co-plot: A graphic display method for geometrical representations of MCDM. *European Journal of Operational Research,* 125**,** 670-678.

ROSEN, SCHAFFNIT, C. & PARADI, J. C. 1998. Marginal rates and two dimensional level curves in DEA. *Journal of Productivity Analysis,* 9**,** 205-232.

SEXTON, T. R., SILKMAN, R. H. & LOGAN, A. J. 1986. Data Envelopment Analysis: Critique and extensions. *In:* SILKMAN, H. (ed.) *Measuring efficiency: An assessment of data envelopment analysis.* San Francisco: Jossey-Bass Editor.

SOARES DE MELLO, J.C.C.B.; ANGULO-MEZA, L. ; LACERDA, F.G. 2012. A DEA model with a non discritionary variable for Olympic evaluation. Pesquisa Operacional, 32, 21-29.

SOARES DE MELLO, J. C. C. B., LINS, M. P. E. & GOMES, E. G. 2002. Construction of a smoothed DEA frontier. *Pesquisa Operacional,* 28**,** 183-201.

STEWART, T. J. 1996. Relationships between Data Envelopment Analysis and Multicriteria Decision Analysis. *Journal of the Operational Research Society,* 47**,** 654-665.

SUZUKI, S., NIJKAMP, P., RIETVELD, P. & PELS, E. 2010. A distance friction minimization approach in data envelopment analysis: A comparative study on airport efficiency. *European Journal of Operational Research,* 207**,** 1104-1115.

THANASSOULIS, E. 2000. The use of data envelopment analysis in the regulation of UK water utilities: Water distribution. *European Journal of Operational Research,* 126**,** 436-453.