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Interfaces with Other Disciplines

A multi-criteria approach to the *h*-index

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ABSTRACT

The *h*-index was proposed in 2005, to objectively measure the impact and relevance of individuals' scientific output, based on their number of publications and citations. Since then, many works have studied the index's advantages and limitations, in various contexts and viewpoints. Still, we contribute to such a vast literature, by adopting a multi-criteria perspective to this matter. More precisely, we study the *h*-index in light of the fundamental axioms of coherence. We show that the number of publications and the number of total citations alone do not compose a coherent criteria family for the *h*-index, because it does not follow the axiom of exhaustivity. Thus, in this paper, we study examples from the literature to propose a suitable third criterion for the *h*-index, specifically, the form of the citation distribution among publications, which is less obvious than the other two. We verify that, with all three criteria, the criteria family that forms the *h*-index becomes coherent.

1. Introduction

Measurement and evaluation of scientific work are an increasing concern in the scientific community (Coccia, 2008; Huber, 2002). From this standpoint, Hirsch (2005) proposed the *h*-index to objectively measure the impact and relevance of scientific output, based on an individualś publications and citations. As consequence of its importance and simplicity, the *h*-index became very popular (Franceschini & Maisano, 2010; Korhonen, Tainio, & Wallenius, 2001). For a recent application of the *h*-index and other bibliometric indicators, see Laengle et al. (2017), for instance.

From the literature perspective, this popularity yielded many studies on the *h*-indexś limitations, such as Antonakis and Lalive (2008), Bornmann and Daniel (2007, 2009), Braun, Glañzel, and Schubert (2006), Castillo, Donato, and Gionis (2007), Egghe (2010), Harzing and Van der Wal (2008), Norris and Oppenheim (2010), Panaretos and Malesios (2009), Thompson et al. (2009), among others. Other studies propose adjustments to the original index, such as Burrell (2007), Franceschini and Maisano (2010), Schreiber (2008), Sidiropoulos, Katsaros, and Manolopoulos (2007), or entirely new variants, such as Alonso, Cabrerizo, Herrera-Viedma, and Herrera (2010), Anderson, Hankin, and Killworth (2008), Batista, Campiteli, and Kinouchi (2006) Bornmann, Mutz, and Daniel (2008), Cabrerizo, Alonso, Herrera-Viedma, and Herrera (2010), Egghe (2006a), Egghe and Rousseau (2008), Jin (2006), Jin, Liang, Rousseau, and Egghe (2007), Kosmulski (2006, 2007), Tol (2009), Vinkler (2009), Wu (2010), Zhang (2009). For reviews on the *h*-index and its variants, see Alonso, Cabrerizo, Herrera-Viedma, and Herrera (2009), Egghe, (2010), Norris and Oppenheim (2010), Ruscio, Seaman, D'Oriano, Stremlo, and Mahalchik (2012) and Schreiber, Malesios, and Psarakis (2011).

The present work studies some *h*-index properties, using a different perspective, more specifically, the multi-criteria perspective. In this paper, we analyse the *h*-index as a criteria family and formalize which criteria compose the index, so that they form a coherent criteria family (Roy & Bouyssou, 1993). Certain studies, such as Benevenuto, Laender, and Alves (2016), Egghe (2010) and Franceschini and Maisano (2010), have mentioned that the *h*-index combines two aspects. Particularly, Egghe (2010) affirms: "The *h*-index is a single simple measure that combines papers (an aspect of quantity) and citations (an aspect of quality i.e., impact)."

However, in this paper, we show that the number of publications and citations alone do not follow the axiom of exhaustivity (Roy & Bouyssou, 1993). We explain that the *h*-index requires three criteria; the third, less obvious than the other two, being the form of the citation distribution among publications. Although Hirsch (2005) already mentioned that the relation between *h* and the total number of citations depends on the particular distribution, herein we show that the distribution form must be considered a formal criterion and elucidate any eventual confusion regarding this matter.

In this study, we do not adopt a statistical perspective towards the *h*-index, as did Glänzel (2006) and Hirsch (2007), for instance. Other studies, such as Anderson et al. (2008),

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Lando and Bertolli-Barsotti (2014), Prathap (2014), and Zhang (2009), have even proposed new *h*-type indexes to improve its sensitivity to the form of the citation distribution. However, we do not assume any statistical distribution for citations herein, because this would conflict with the formalization of the criteria family. First, such an assumption would imply in loss of generality. Though most importantly, we would drive the attention away from the citation distribution and towards the parameters used. Depending on the probability model, we might even disregard the form of the citation distribution for most cases. In fact, according to Burrell (2013) "a good estimate of the *h*-index can often be obtained knowing only the number of publications and the number of citations". In line with this, Bertolli-Barsotti and Lando (2015) develop a formula for the *h*-index using the number of publications and the mean of citations, whereas the maximum number of citations acts as a mere adjustment term. For these reasons, in this paper, we do not make any assumptions regarding the citation distribution form and show that this should be a formal criterion.

Our study does not affect the *h*-index's calculations or results. Our purpose is to analyse the index, using tools and concepts of Multiple Criteria Decision Making (MCDM). Billaut, Bouyssou, and Vincke (2010), for instance, adopted a similar perspective with regard to the Shanghai ranking. Moreover, there are several works on the *h*-index and its variants, such as Bouyssou and Marchant (2014), Deineko and Woeginger (2009), Hwang (2013), Marchant (2009), Miroiu (2013), Quesada (2009, 2010, 2011a, 2011b), and Woeginger, 2008a, 2008b), which adopted a multi-criteria approach, however, with a different perspective. These studies proposed axioms that described desirable properties of scientific impact indexes, and analysed which bibliometric indexes satisfy each condition. Despite their differences, the purpose of both the present work as well as these axiomatic studies is to help improve understanding of this popular bibliometric index, and assist universities, journals, and other potential users of bibliometric indicators in their decisions.

The next section briefly explains the *h*-index and highlights some disadvantages from the literature, related to the present study. The third section presents a literature review on axiomatic analyses of the *h*-index and its variants. The following section studies the *h*-index with regard to the fundamental axioms of coherence. Finally, the last section presents concluding remarks.

2. Considerations on the *h*-index

The *h*-index (Hirsch, 2005) is a simple measure of impact and relevance of an individual's scientific research output, combining the number of publications and the number of citations. By definition, a scientist has index *h*, if *h* publications have at least *h* citations each, while the other publications have no more than *h* citations each. The *h* publications cited at least *h* times compose the so-called *h*-core.

The *h*-index may be considered robust (Orbay, Karamustafaoglu, & Oner, 2007), because of two main reasons. First, once a certain publication has entered the *h*-core, an increase in its number of citations does not affect the *h*-index. Moreover, publications with few citations do not influence the *h*-index either.

However, this robustness may be considered weak sensitivity to highly cited papers, and therefore, be interpreted as a disadvantage (Egghe, 2010; Schreiber, 2010). It relates to the index's condition of non-strict monotony, in which an increase/decrease in the number of citations per paper is not associated with an increase/decrease in the *h*-index (Franceschini, Galetto, & Maisano, 2007). Moreover, the unfavourable robustness may also be related to the lack of consistency between rankings at different levels of aggregation (Waltman & Van Eck, 2012).

Another disadvantage, related to the present study, is the implicit parameter, which the *h*-index is subject to (Waltman & Van Eck, 2012). Although Hirsch (2005) affirmed that the *h*-index had the advantage of not depending on an arbitrary parameter, Waltman and Van Eck (2012) disagree. According to the latter, the requirement of having at least *h* citations for each publication in the *h*-core hides an arbitrary parameter. The index could have required $2h$, $h/2$ or any other number of citations, which would lead to an entirely different index and ranking.

3. Existing axiomatic analyses of the *h*-index

As previously mentioned, many studies have pointed several disadvantages of the *h*-index, and proposed corrections or new variants. However, some of these studies may be considered myopic because they focus on the index's flaws and ignore other features (Marchant, 2009). This approach may lead to new weaknesses that the previous index did not present.

This is why certain studies provided thorough analyses of the *h*-index, some of which included many other bibliometric rankings as well. Marchant (2009), for instance, analysed a set of properties that entirely characterizes several different rankings, including the *h*-index. Seeing that there is no true ranking, this study presented the axioms that each one satisfies, in order to help select the most suitable ranking for different situation.

Other studies also provided axiomatic analysis on the *h*-index, such as Deineko and Woeginger (2009), Hwang (2013) and Miroiu (2013), Quesada (2009, 2010, 2011a, 2011b), Woeginger, 2008a, 2008b). However, they present limitations, because they use axioms that are not easy to interpret and they analyse only one index, instead of its entire family, e.g., index *h*' defined as 100 times the *h*-index (Bouyssou & Marchant, 2014).

This is why Bouyssou and Marchant (2014) also used an axiomatic approach, though striving to overcome such limitations. Besides analysing the entire index family, the authors also contributed to the literature by studying indexes-induced rankings, as well as indexes. Compared to Marchant (2009), they enlarged the list and also analysed indexes, not only rankings. Bouyssou and Marchant (2014) proposed axioms that are valid, easily interpreted and common to several indexes and rankings. However, the analysis of the *h*-index required the definition of many new exclusive conditions.

Although the present paper is based on axiomatic analyses as well, herein we consider the fundamental axioms of coherence (Roy & Bouyssou, 1993), instead of axioms developed from desirable properties for bibliometric indicators, as explained in the next section.

4. A multi-criteria approach to the *h*-index

Considering the fundamental axioms of coherence, herein we show that the *h*-index is not based on a criteria family that depends solely on the total number of publications (p) and the total number of citations (N_c). Then, we introduce the third criterion and analyse the *h*-index as a criteria family.

4.1. The h-index in light of the coherence axioms

Previous works, such as Franceschini and Maisano (2010), identified that, although based on the number of publications and citations, the *h*-index could evaluate scientific output in contrast to such measures. In other words, it is possible for a researcher to have a smaller *h*-index, and yet more citations and publications. Table 1, extracted from Franceschini and Maisano (2010), showing the rank frequency distribution for two scholars, consists in an example of such incoherence.

Table 1
h-index in contrast to the number of publications and total citations.

Paper rank	Scholar *a* Citations per paper	Scholar *b* Citations per paper
1	12	20
2	7	15
3	6	10
4	4	3
5	3	3
6	2	3
7	1	3
8	1	3
9	1	3
10		3
11		3
N_c	37	69
p	9	11
h	4	3

N_c: Total number of citations.
p: Number of publications.
h: *h*-index.

It is worth highlighting that this example is not unique. On the contrary, it is quite easy to find examples where the *h*-index evaluates researchers in a reverse order, when compared to their number of publications and citations. This means that if the *h*-index were based on a criteria family that depended only on the total number of citations and on the number of publications, this two criteria family would be incoherent.

We show this, using the fundamental axioms of coherence imposed to any criteria family: exhaustivity, cohesion and non-redundancy (Roy & Bouyssou, 1993). Specifically, it is the axiom of exhaustivity that would not be verified if the *h*-index were based on a two criteria family.

Formally, Roy and Bouyssou (1993) define the axiom of exhaustivity as (1), where j is a criterion in the criteria family F; a, b and c are actions under evaluation; and $g_j(a)$ and $g_j(b)$ represent the performance of a and b, respectively, with regard to criterion j. In the *h*-index context, j represents each criterion, i.e., number of publications and total number of citations; and a, b and c represent researchers under evaluation.

If $\forall j \in F, g_j(b) = g_j(a)$, hence for any c:

$$\begin{aligned} &cHb \to cHa,\ \forall H \in \{I, P, Q, R, \sim, \succ, S\} \\ &bH'c \to aH'c, \forall H' \in \{I, P, Q, R, \sim, \succ, S\} \end{aligned} \tag{1}$$

In (1), H and H' represent binary relations which may be of type $I, P, Q, R, \sim, \succ$ or S. I means that both actions are indifferent, P means that the first action is strictly preferable to the second, Q means that the first is weakly preferable to the second (somewhere between indifference and strict preference), R means absence of a clear and positive relation between the two actions (incomparability), $\sim$ means that the actions are either indifferent or incomparable (I or R), $\succ$ means that the first action is either strictly or weakly preferable to the second (P or Q), and S means existence of a clear and positive relation, which may be of strict or weak preference, or indifference (P, Q or I), indistinctively.

As a consequence of definition (1), we have the statement (2), which may be regarded as a particular case of (1), where action c is identical to action b (Roy & Bouyssou, 1993).

$$\text{If}\, \forall j \in F, g_j(b) = g_j(a)\, \text{hence}\, b\ I\ a \tag{2}$$

In words, if two actions a and b perform the same in all criteria j in the criteria family F, then these actions should be considered indifferent to each other.

However, it is possible to find examples where two researchers are equally evaluated in both criteria, number of publications and total number of citations, and yet have different *h*-indexes.

Table 2
The *h*-index's lack of exhaustivity.

Paper rank	Egghe	Egghe'	Egghe″
1	47	57	37
2	42	51	33
3	37	45	29
4	36	43	29
5	21	27	21
6	18	23	21
7	17	21	20
8	16	19	19
9	16	18	18
10	16	17	18
11	15	14	17
12	13	11	16
13	13	10	16
14	13	9	15
15	13	8	15
16	12	6	15
17	12	5	14
18	12	4	13
19	12	3	13
20	11	1	13
N_c	392	392	392
p	20	20	20
h	13	11	15

N_c: Total number of citations.
p: Number of publications.
h: *h*-index.

Table 1 shows a more extreme example for this situation, where the *h*-index is in reverse order with the number of publications and citations. However, Table 1 does not follow the statement's hypothesis exactly. For that, we show in Table 2, partially extracted from Egghe (2006b), the rank frequency distribution for Egghe's citations, from January 2006, as well as for two fabricated variations (Egghe' and Egghe″), with the same number of publication (p) and the same total number of citations (N_c), yet different *h*-indexes.

The example from Table 2 shows that if the *h*-index were based on a criteria family F, whose criteria (j) are the total number of citations and the number of publications, we would have $g_j(a) = g_j(b) = g_j(c)\ \ \forall j \in F$, where a represents Egghe, b represents Egghe', and c represents Egghe″. However, a, b and c are not indifferent, instead $c\ P\ a\ P\ b$, as shown in Table 1.

We may find countless examples with the same characteristics, i.e., same p and N_c, yet different *h*-indexes. Given one researcher with p publications, N_c citations and an *h*-index, it is possible to find smaller *h*-indexes, with the same p and N_c, by varying the distribution of citations inside the *h*-core and leaving one or more publication with less than h citations. It is also possible to find higher *h*-indexes, with the same p and N_c, by redistributing some of the citations from publications inside the core, to publications outside the core. In other words, if the h index were based on a criteria family of p and N_c, we could easily observe that the exhaustivity axiom would be violated, thus rendering the criteria family incoherent.

According to Roy and Bouyssou (1993), this problem could be solved by redefining the existing criteria, or in the simplest case, with an extra criterion. As it does not seem possible to redefine either criteria, number of publications or number of citations, so that exhaustitvity is ensured, we define, in the next section, a third criterion for the *h*-index.

4.2. About the third criterion

In this section, we explain that a suitable third criterion relates to the citation distribution, which is less obvious than the number of total citations and number of publications.

Table 3
Number of citations for each publication.

Paper rank	Egghe	Rousseau	Leydesdorff	White	Martin	Van Raan
1	47	25	79	128	156	108
2	42	18	32	106	74	51
3	37	18	26	103	52	49
4	36	16	24	45	38	41
5	21	16	23	37	35	35
6	18	15	22	28	33	32
7	17	15	19	22	33	31
8	16	14	17	21	30	30
9	16	13	17	20	29	25
10	16	13	16	15	28	25
11	15	13	15	14	24	23
12	13	13	13	14	23	22
13	13	13	13	12	22	22
14	13	12	13	12	20	21
15	13	12	11	12	19	20
16	12	12	11	12	18	19
17	12		11	11	16	19
18	12		10	10	16	19
19	12		10	8	16	19
20	11		9	6	16	18
21				5	14	18
22				5	14	17
23				5	11	17
24				5	9	17
25				4	9	15
26				4	9	14
27					6	14
28					4	14
N_c	392	238	391	664	774	755
p	20	16	20	26	28	28
h	13	13	13	12	16	19

N_c: Total number of citations.
p: Number of publications.
h: h-index.

For this purpose, we use the example in Table 3, extracted from Egghe's (2006b) citation database, from January 2006, also studied in Rubem, Moura, and Soares de Mello (2015), for instance. Table 3 presents the rank frequency distribution for six authors, namely Leo Egghe, Loet Leydesdorff, Ronald Rousseau, Howard D. White, Ben Martin, and Anthony F. J. Van Raan. Table 3 also presents the number of publications, total citations, and the h-index, for each author.

We may observe that Rousseau presents the least number of total citations and publications, though his h-index is higher than White's. The reason for this is that White's citations are considerably concentrated – half of all citations are concentrated on three publications, whereas Rousseau's citations are very well distributed. Moreover, Rousseau's h-index is the same as Egghe's and Leydesdorff's, who present 64% more citations than Rousseau, though with higher concentration. Particularly, 20% of Leydesdorff's citations are attributed to a single publication.

To further illustrate this less explicit criterion, we redistributed citations from Table 3 to maximize the h-index, as shown in Table 4. We should highlight that such redistribution is not possible for actual researchers and their scientific output. In other words, the researchers shown in Table 3 would never have their scientific output as shown in Table 4. The purpose of this modification is simply to help illustrate the importance of the citation distribution criterion for a researcher's h-index.

Comparing Tables 3 and 4, we could observe that citations were dispersed, which led to an increase of the h-index. Clearly, this is not the single citation distribution that leads to this maximum h-index. If Egghe had a single citation for his 20th publication, 19 citations for each publication, from the second to the 19th publication, and 49 citations for the first publication, for instance, his h-index would also be 19, as in Table 4.

From these examples, we could observe that the citation distribution is an essential aspect for the h-index criteria family. We show that the h-index depends on three different and independent aspects, i.e., number of publications, number of citations, and how these citations are distributed among publications, which should be defined as separate criteria. Thus, the h-index requires three criteria, the third being the form of the citation distribution, herein, referred to as $c\sim$. In other words, for a given form of citation distribution, the h-index varies depending on the number of citations and publications.

For a given number N_c of citations and p publications, the h-index will be higher when more n_c citations are equally distributed among $\sqrt{n_c}$ publications, where $n_c \le N_c$ and $\sqrt{n_c} \le p$. In this case, all $\sqrt{n_c}$ publications would have at least $\sqrt{n_c}$ citations each, which leads to an h-index of $\sqrt{n_c}$, according to its definition. The other $N_c - n_c$ citations could have any distribution. In Egghe's redistributed example from Table 4, there were 19 citations equally distributed among 19 publications, whereas the other 31 citations could have been attributed to any publication, either concentrated on this first, or somehow distributed among various publications, without affecting his h-index.

Consequently, the highest h-index for a researcher with N_c citations and $p \ge \sqrt{N_c}$ publications is $\sqrt{N_c}$, if all citations are equally distributed among $\sqrt{N_c}$ publications. More precisely, it is the highest natural number $h \le \sqrt{N_c}$. This situation is illustrated in Table 4 for all authors. For Egghe, $\sqrt{N_c} = \sqrt{392} \cong 19.8$, thus Egghe's highest h-index with the same number of citations and the same

Table 4
Redistribution of citations to maximize the *h*-index.

Paper rank	Egghe	Rousseau	Leydesdorff	White	Martin	Van Raan
1	20	15	20	26	28	28
2	20	15	20	26	28	28
3	20	15	20	26	28	28
4	20	15	20	26	28	28
5	20	15	20	26	28	28
6	20	15	20	26	28	28
7	20	15	20	26	28	28
8	20	15	20	26	28	28
9	20	15	20	26	28	28
10	20	15	20	26	28	28
11	20	15	20	26	28	28
12	20	15	20	26	28	28
13	20	15	20	26	28	28
14	20	15	20	26	28	28
15	20	15	20	26	28	28
16	20	13	20	26	28	28
17	20		20	26	28	28
18	20		20	26	28	28
19	20		20	26	28	28
20	12		11	26	28	28
21				26	28	28
22				26	28	28
23				26	28	28
24				26	28	28
25				26	28	28
26				14	28	27
27					28	27
28					18	1
N_c	392	238	391	664	774	755
p	20	16	20	26	28	28
h	19	15	19	25	27	27

N_c: Total Number of Citations.
p: Number of Publications.
h: h-index.

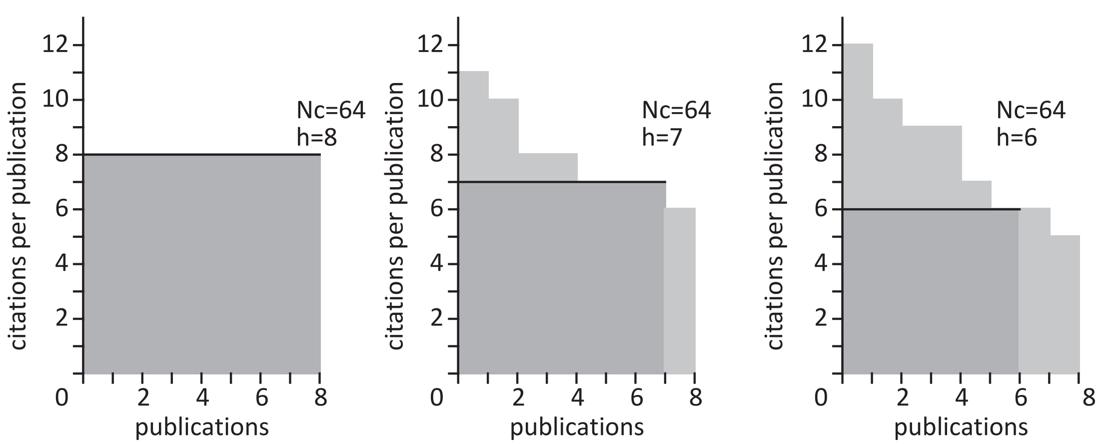

Fig. 1. Illustrative examples of different citation distribution.

number of publications is $h=19$, as shown in Table 4. Similarly, for Martin, $\sqrt{N_c} = \sqrt{774} \cong 27.8$, thus his highest h-index, is $h=27$, which is also presented in Table 4.

When $p < \sqrt{N_c}$, which is unrealistic according to Hirsch (2005), and not true for any author from Tables 2 and 3, the maximum h-index would be p, if $n_c = p^2$ citations were equally distributed among the p publications. In this case, the other $N_c - n_c$ citations could also have any distribution.

To illustrate these definitions graphically, let us suppose there are a total of $N_c = 64$ citations and $p = 8$ publications, as shown in all cases of Fig. 1. The dark grey area represents the n_c citations that are equally distributed, which affect the h index, and the light grey area, in the second and third illustrations, represent the $N_c - n_c$ citations that do not affect the index.

The highest possible h index occurs when the total number of citations ($N_c = 64$) are equally distributed among $\sqrt{N_c} = 8$ publications. This is why the "ideal distribution" for Land and Bertoli-Barsotti's (2014) h-type index, for instance, is squared-form. The second highest index occurs when $n_c = (\sqrt{N_c} - 1)^2 = 49$ citations equally distributed among $\sqrt{n_c} = \sqrt{N_c} - 1 = 7$ publications. Then, the following highest index occurs with $n_c = (\sqrt{N_c} - 2)^2 = 36$ citations equally distributed among $\sqrt{n_c} = \sqrt{N_c} - 2 = 6$ publications, etc.

4.3. A three criteria family

In this section, we define the criteria family that forms the h-index, in order to overcome the problem of incoherence shown in previous sections.

Defined as a criteria family of N_c, p and $c\sim$, it is not possible to find two cases (a and b) that are equally evaluated in all three criteria, i.e., presents the same number of publications, the same number of citations, and the same distribution of citations among publications, with different h-indexes. When all three criteria are fixed, the number of citations for each and every publication is also fixed, and, consequently, the number h of publications that has at least h citations each, while the other publications have no more than h citations each. By definition, the h-index for a and b would be the same, thus, $a\ I\ b$. Even if we consider a third case (c), the relation between a and c would be exactly the same as the relation between b and c, as shown in (1), because the h-index of a and b are identical. In other words, this criteria family follows the axiom of exhaustivity.

The second fundamental axiom of coherence is cohesion, formally defined as (3), divided in two parts, (3.1) and (3.2), (Roy & Bouyssou, 1993), where all variables are the same as in (1).

$$\begin{aligned}&\text{For}\, g_j\left(b^k\right) = g_j(b)\,\text{and}\, g_j(a) = g_j(a_k),\ \forall j \neq k\\&\text{If}\, g_k\left(b^k\right) \geq g_k(b)\,\text{and}\, g_k(a) \geq g_k(a_k),\end{aligned} \tag{3}$$

where at least one of the inequality restrictions are strict, then:

$$\begin{aligned}&b\ P\ a \to b^k\ P\ a_k,\\&b\ Q\ a \to b^k \succ a_k,\\&b\ I\ a \to b^k\ S\ a_k\end{aligned} \tag{3.1}$$

Observation: grouped relations ($\succ$,S) could be replaced by one of the fundamental relations they could imply, i.e., $\succ$ could be replaced by P or Q, and S could be replaced by $P,\ Q$ or I.

On the other hand, if $g_k(b) = g_k(a)$ and $b^k\ I_k\ a_k$, then $\forall\, H \in \{I, Q, P, R, \sim, \succ, S\}$:

$$\begin{aligned}&b\ H\ a \to b^k\ H\ a_k,\\&a\ H\ b \to a_k\ H\ b^k.\end{aligned} \tag{3.2}$$

In (3.1), b^k represents an improvement of b in criterion k and/or a_k represents a deterioration of a in criterion k, whereas all the rest remains the same. In this case, if $b\ P\ a$, then $b^k\ P\ a_k$. In other words, if researcher b has a higher h-index than researcher a ($b\ P\ a$), and if b has an increase in the number of citations, for instance, whereas all the other attributes remain the same, it is certain that the h-index for b will continue to be higher ($b^k\ P\ a_k$). Similarly, if researchers a and b have the same h-index ($b\ I\ a$) and if b has an increase in the number of citations, for instance, then the h-index for b will be higher than or the same as the h-index for a *($b\ S\ a$)*. These relations from (3.1) remain valid for any other criterion of the criteria family.

In (3.2), a and b perform the same in a certain criterion and a slight positive change in b with respect to this criterion (b^k) remains indifferent to a slight negative change in a with respect to the same criterion (a_k), then whatever relation between a and b remains unaltered between b^k and a_k.

In the h-index context, although the relations in (3.2) are true, they do not make sense. For example, let us suppose that two researchers (a and b) present the same number of publications. It is not possible that b experiences a positive change with respect to the number of publications (b^k) and/or a experiences a negative change with respect to the same criterion (a_k), and still have the same number of publications, i.e., it wouldnt́ be possible to have $b^k\ I_k\ a_k$.

This does not mean that the h-index does not follow the relations in (3.2). This simply means that if both researchers remained indifferent to each other with regard to a certain criterion ($b^k\ I_k\ a_k$), then no modification would have occurred with regard to such criterion, as shown in (4).

$$g_k\left(b^k\right) = g_k(b)\,\text{and}\, g_k(a) = g_k(a_k) \tag{4}$$

Seeing that we originally defined in (3) $g_j(b^k) = g_j(b)$ and $g_j(a) = g_j(a_k),\ \forall j \neq k$, together with (4), we have $g_j(b^k) = g_j(b)$ and $g_j(a) = g_j(a_k),\ \forall j \in F$. From the exhaustivity axiom in (2), this implicates that $b^k\ I\ b$ and $a\ I\ a_k$. Now, it becomes simple to check that the h-index follows (3.2): if researchers b^k and b have the same h-index (l) and researchers a and a_k have the same h-index (m), whatever relation holds between the h-index of b and a (i.e., between l and m) holds between the h-index of b^k and a_k (also l and m).

Seeing that the h-index follows (3.1) and (3.2), the criteria family follows the axiom of cohesion.

The third and last fundamental axiom of coherence is non-redundancy. A criteria family F is non-redundant if the subtraction of any criterion h from F renders the criteria sub-family $F\backslash\{h\}$ either non-exhaustive or non-cohesive.

We have already shown in the previous section that the subtraction of the citation distribution criterion renders the h-index non-exhaustive. Similarly, if we ignore a researchers̀ number of publications or number of citations, the h-index would also become non-exhaustive. After all, for a fixed number of citations and distribution of citations, there could be different h-indexes, depending on the number of publications; and for a fixed number of publications and distribution of citations, there could be different h-indexes, depending on the number of total citations.

As a result, the subtraction of any criterion would render the criteria family non-exhaustive. Thus, by definition, the h-index is based on a non-redundant criteria family. Seeing that it is also exhaustive and cohesive, such three criteria family is cohesive.

5. Conclusions

The h-index measures the scientific output of researchers, according to their number of publications and number of citations. Possibly due to its simplicity and importance, this bibliometric index became very popular, yielding many studies on its strengths and limitations. Still, this paper contributes to the existing literature, by adopting a different perspective. Herein, we have used a multi-criteria approach to study the h-index as a criteria family, in light of the fundamental axioms of coherence, namely, exhaustivity, cohesion and non-redundancy.

We find that if the h-index were only a two-criterion family, depending on the number of publications and citations, it would lack exhaustivity, and therefore coherence. Thus, we acknowledge a third, less explicit, criterion, specifically, the form of the citation distribution, to overcome such problem.

The elicitation of this third criterion does not affect any of the h-indexs̀ advantages or disadvantages, widely studied in the literature. However, elucidating this matter might contribute to discussions. For instance, this third criterion explains why an increase in the number of publications and/or citations may not lead to a higher index. It is also related to the unfavourable robustness identified in Egghe (2010) and Schreiber (2010), as well as to the non-strict monotony explained in Franceschini et al. (2007). Similarly, the h-index is not suitable for aggregations, as pointed out in Waltman and Van Eck (2012) and Rubem et al. (2015), because it does not depend only on the aggregated amount of citations and publication, but also on the aggregated citation's distribution. Moreover, Waltman and Van Eck's (2012) hidden arbitrary parameter is intimately related to our third criterion. The fact that Hirsch (2005) required that each publication in the h-core had at least h citations each, instead of $h/2$ or $2\,h$, also defines what type of citations distribution is preferred. If the index had a different parameter, then the preferred citation distribution would also be different.

Future works could provide similar multi-criteria analyses to other *h*-type indexes and their rankings.

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References

Alonso, S., Cabrerizo, F. J., Herrera-Viedma, E., & Herrera, F. (2009). *h*-index: A review focused on its variants, computation and standardization for different scientific fields. *Journal of Informetrics, 3*, 273–289.
Alonso, S., Cabrerizo, F. J., Herrera-Viedma, E., & Herrera, F. (2010). hg-Index: A new index to characterize the scientific output of researchers based on the h- and g-indices. *Scientometrics, 82*(2), 391–400.
Anderson, T., Hankin, R., & Killworth, P. (2008). Beyond the Durfee square: Enhancing the h- index to score total publication output. *Scientometrics, 76*(3), 577–588.
Antonakis, J., & Lalive, R. (2008). Quantifying scholarly impact: IQp versus the Hirsch h. *Journal of the American Society for Information Science and Technology, 59*(6), 956–969.
Batista, P. D., Campiteli, M. G., & Kinouchi, O. (2006). Is it possible to compare researchers with different scientific interests. *Scientometrics, 68*(1), 179–189.
Benevenuto, F., Laender, A. H. F., & Alves, B. L. (2016). The H-index paradox: Your coauthors have a higher *h*-index than you do. *Scientometrics, 106*, 469–474. doi:10.1007/s11192-015-1776-2.
Bertolli-Barsotti, L., & Lando, T. (2015). On a formula for the *h*-index. *Journal of Informetrics, 9*(4), 762–776.
Billaut, J.-C., Bouyssou, D., & Vincke, P. (2010). Should you believe in the Shanghai ranking? An MCDM view. *Scientometrics, 84*(1), 237–263. doi:10.1007/s11192-009-0115-x.
Bornmann, L., & Daniel, H. D. (2007). What do we know about the h index. *Journal of the American Society for Information Science and Technology, 58*(9), 1381–1385.
Bornmann, L., & Daniel, H.-D. (2009). The state of h index research. Is the h index the ideal way to measure research performance. *EMBO Reports, 10*(1), 2–6.
Bornmann, L., Mutz, R., & Daniel, H.-D. (2008). Are there better indices for evaluation purposes than the *h* index? A comparison of nine different variants of the *h* index using data from biomedicine. *Journal of the American Society for Information Science and Technology, 59*(5), 830–837.
Bouyssou, D., & Marchant, T. (2014). An axiomatic approach to bibliometric rankings and indices. *Journal of Informetrics, 8*(3), 449–477.
Braun, T., Glänzel, W., & Schubert, A. (2006). A Hirsch-type index for journals. *The Scientist, 69*(1), 169–173.
Burrell, Q. L. (2007). Hirsch index or Hirsch rate? Some thoughts arising from Liang's data. *Scientometrics, 73*(1), 19–28.
Burrell, Q. L. (2013). The *h*-index: A case of the tail wagging the dog. *Journal of Informetrics, 7*(4), 774–783.
Cabrerizo, F. J., Alonso, S., Herrera-Viedma, E., & Herrera, F. (2010). q2-Index: Quantitative and qualitative evaluation based on the number and impact of papers in the Hirsch core. *Journal of Informetrics, 4*(1), 23–28.
Castillo, C., Donato, D., & Gionis, A. (2007). Estimating number of citations using author reputation. In N. Ziviani, & R. Baeza-Yates (Eds.), *String processing and information retrieval* (pp. 107–117). Berlin, Heilberg: Springer.
Coccia, M. (2008). Measuring scientific performance of public research units for strategic change. *Journal of Informetrics, 2*, 183–194.
Deineko, V. G., & Woeginger, G. J. (2009). A new family of scientific impact measures: The generalized Kosmulski-indices. *Scientometrics, 80*(3), 819–828.
Egghe, L. (2006a). An improvement of the *h*-index: The g-index. *ISSI Newsletter, 2*(1), 8–9.
Egghe, L. (2006b). Theory and practise of the *g*-index. *Scientometrics, 69*(1), 131–152.
Egghe, L. (2010). The Hirsch index and related impact measures. *Annual Review of Information Science and Technology, 44*, 65–114.
Egghe, L., & Rousseau, R. (2008). An *h*-index weighted by citation impact. *Information Processing and Management, 44*(2), 770–780.
Franceschini, F., Galetto, M., & Maisano, D. (2007). *Management by measurement: designing key indicators and performance measurement systems*. Berlin: Springer-Verlag.
Franceschini, F., & Maisano, D. A. (2010). Analysis of the Hirsch index's operational properties. *European Journal of Operational Research, 203*(2), 494–504.
Glänzel, W. (2006). On the *h*-index – A mathematical approach to a new measure of publication activity and citation impact. *Scientometrics, 67*(2), 315–321.
Harzing, A. W., & Van der Wal, R. (2008). Google Scholar as a new source for citation analysis. *Ethics in Science and Environmental Politics, 8*(11), 61–73.
Hirsch, J. E. (2005). An index to quantify an individual's scientific research output. *Proceedings of the National Academy of Sciences, 102*(46), 16569–16572.
Hirsch, J. E. (2007). Does the h index have predictive power. *Proceedings of the National Academy of Sciences, 104*(49), 19193–19198.
Huber, M. T. (2002). Faculty evaluation and the development of academic careers. *New Directions for Institutional Research, 114*, 73–83.
Hwang, Y.-A. (2013). An axiomatization of the Hirsch-index without adopting monotonicity. *Applied Mathematics & Information Sciences. An International Journal, 7*(4), 1317–1322.
Jin, B. (2006). H-index: An evaluation indicator proposed by scientist. *Science Focus, 1*(1), 8–9.
Jin, B., Liang, L., Rousseau, R., & Egghe, L. (2007). The R- and AR-indices: Complementing the h-index. *Chinese Science Bulletin, 52*(6), 855–863.
Korhonen, P., Tainio, R., & Wallenius, J. (2001). Value efficiency analysis of academic research. *European Journal of Operational Research, 130*(1), 121–132.
Kosmulski, M. (2006). A new Hirsch-type index saves time and works equally well as the original h-index. *ISSI Newsletter, 2*(3), 4–6.
Kosmulski, M. (2007). MAXPROD – A new index for assessment of the scientific output of an individual, and a comparison. *Cybermetrics, 11*(1), 1–5.
Laengle, S., Merigó, J. M., Miranda, J., Slowinski, R., Bomze, I., Borgonovo, E., Dyson, R. G., Oliveira, J. F., & Teunter, R. (2017). Forty years of the European journal of operational research: A bibliometric overview. *European Journal of Operational Research, 262*(3), 803–816. doi:10.1016/j.ejor.2017.04.027.
Lando, T., & Bertolli-Barsotti, L. (2014). A new bibliometric index based on the shape of the citation distribution. *PlosOne, 9*(12), e115962.
Marchant, T. (2009). An axiomatic characterization of the ranking based on the h-index and some other bibliometric rankings of authors. *Scientometrics, 80*(2), 327–344.
Miroiu, A. (2013). Axiomatizing the Hirsch index: Quantity and quality disjoined. *Journal of Informetrics, 7*(1), 10–15.
Norris, M., & Oppenheim, C. (2010). The h-index: A broad review of a new bibliometric indicator. *Journal of Documentation, 66*(5), 681–705.
Orbay, M., Karamustafaoglu, O., & Oner, F. (2007). What does Hirsch index evolution explain us? A case study: Turkish Journal of Chemistry. *Biblios, 27*, 1–5.
Panaretos, J., & Malesios, C. (2009). Assessing scientific research performance and impact with single indices. *Scientometrics, 81*(3), 635–670.
Prathap, G. (2014). The Zynergy-Index and the Formula for the h-Index. *Journal of the Association for Information Science and Technology, 65*(2), 426–427.
Quesada, A. (2009). Monotonicity and the Hirsch index. *Journal of Informetrics, 3*(2), 158–160.
Quesada, A. (2010). More axiomatics for the Hirsch index. *Scientometrics, 82*, 413–418.
Quesada, A. (2011a). Axiomatics for the Hirsch index and the Egghe index. *Journal of Informetrics, 5*, 476–480.
Quesada, A. (2011b). Further characterizations of the Hirsch index. *Scientometrics, 87*, 107–114.
Roy, B., & Bouyssou, D. (1993). *Aide Multicriteŕe a`la dećision: Meťhodes et cas*. Paris: Economica.
Rubem, A. P. S., Moura, A. L., & Soares de Mello, J. C. C. B. (2015). Comparative analysis of some individual bibliometric indices when applied to groups of researchers. *Scientometrics, 102*(1), 1019–1035. doi:10.1007/s11192-014-1428-y.
Ruscio, J., Seaman, F., D'Oriano, C., Stremlo, E., & Mahalchik, K. (2012). Measuring scholarly impact using modern citation-based indices. *Measurement: Interdisciplinary Research and Perspectives, 10*(3), 123–146.
Schreiber, M. (2008). A modification of the h-index: The hm-index accounts for multiauthored manuscripts. *Journal of Informetrics, 2*(3), 211–216.
Schreiber, M. (2010). Twenty Hirsch index variants and other indicators giving more or less preference to highly cited papers. *Annalen der Physik, 522*(8), 536–554.
Schreiber, M., Malesios, C. C., & Psarakis, S. (2011). Categorizing h-index variants. *Research Evaluation, 20*(5), 397–408.
Sidiropoulos, A., Katsaros, D., & Manolopoulos, Y. (2007). Hirsch h-index for disclosing latent facts in citation networks. *Scientometrics, 72*(2), 253–280.
Thompson, D. F., Callen, E. C., & Nahata, M. C. (2009). New indices in scholarship assessment. *American Journal of Pharmaceutical Education, 73*(6), 111.
Tol, R. S. (2009). The h-index and its alternatives: An application to the 100 most prolific economists. *Scientometrics, 80*(2), 317–324.
Vinkler, P. (2009). The π-index: A new indicator for assessing scientific impact. *Journal of Information Science, 35*(5), 602–612.
Waltman, L., & Van Eck, N. J. (2012). The inconsistency of the h-index. *Journal of the American Society for Information Science and Technology, 63*(2), 406–415.
Woeginger, G. J. (2008a). An axiomatic characterization of the Hirsch-index. *Mathematical Social Sciences, 56*, 224–232.
Woeginger, G. J. (2008b). A symmetry axiom for scientific impact indices. *Journal of Informetrics, 2*(4), 298–303.
Wu, Q. (2010). The w-index: A measure to assess scientific impact by focusing on widely cited papers. *Journal of the American Society for Information Science and Technology, 61*(3), 609–614.
Zhang, C.-T. (2009). The e-index, complementing the h-index for excess citations. *PLoS One, 4*(5), e5429. doi:10.1371/journal.pone.0005429.