Pré-Cálculo

Departamento de Matemática Aplicada
Universidade Federal Fluminense

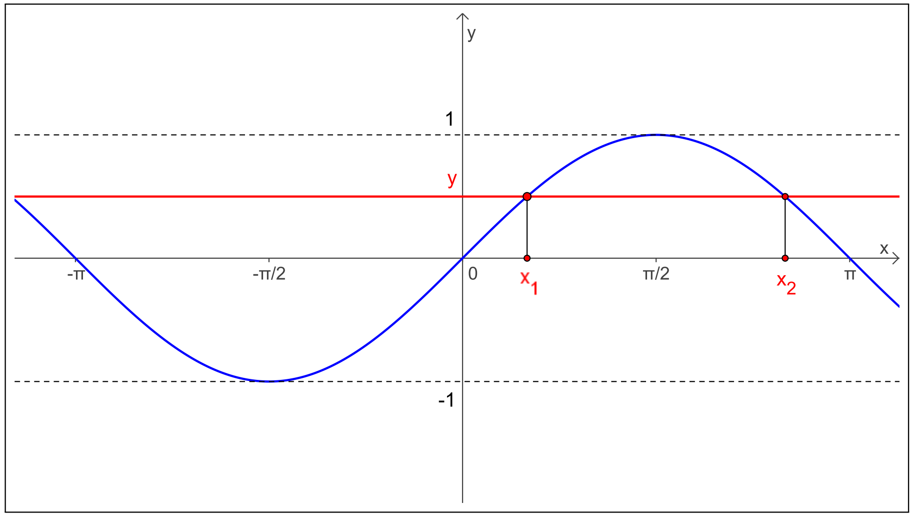
Parte 10

Funções Trigonométricas Inversas

A função arco seno

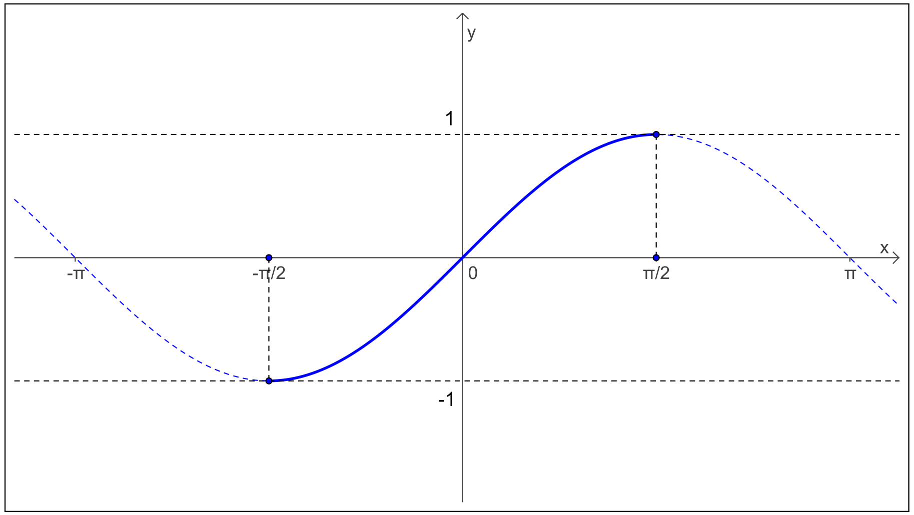
Função Seno

$$\begin{array}{rccl} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & y = f(x) = \text{sen}(x) \end{array}$$ não é inversível, pois não é injetiva.



Função Seno

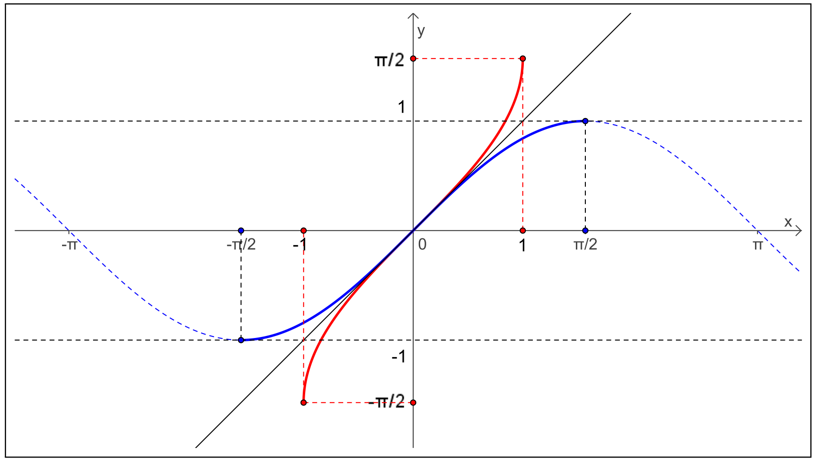
$$f: \begin{array}{rcl} [-\pi/2, +\pi/2] & \to & [-1, +1] \\ x & \mapsto & y = f(x) = \text{sen}(x) \end{array}$$ é inversível, pois é bijetiva.



Função Arco Seno

$$f^{-1}: \begin{array}{rcl} [-1, +1] & \to & [-\pi/2, +\pi/2] \\ x & \mapsto & y = f^{-1}(x) = \text{arcsen}(x) \end{array}$$

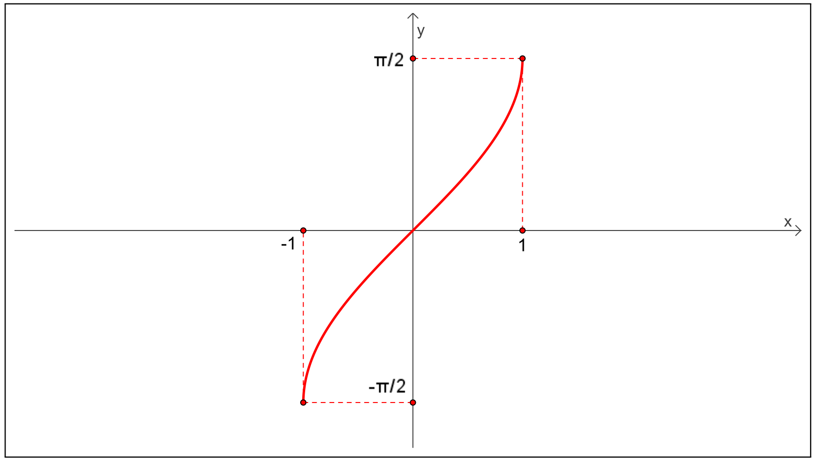
é a função arco seno, ou seja, é a função inversa do seno.



Função arco seno:

$$\boxed{f^{-1}: \begin{array}{rcl} [-1, +1] & \to & [-\pi/2, +\pi/2] \\ x & \mapsto & y = f^{-1}(x) = \text{arcsen}(x) \end{array}}$$

$$\boxed{\text{Para } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y = \text{arcsen}(x), \text{ se e só se, } \text{sen}(y) = x}.$$



Exercício - Calcule:

i) $y = \text{arcsen}(1/2)$

Resolução: Para $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$y = \text{arcsen}\left(\frac{1}{2}\right) \iff \text{sen}(y) = \frac{1}{2} \iff \boxed{y = \frac{\pi}{6}}.$$

ii) $y = 2\,\text{arcsen}(\sqrt{3}/2)$

Resolução: Temos que $y = 2\,\text{arcsen}\left(\frac{\sqrt{3}}{2}\right) \iff \frac{y}{2} = \text{arcsen}\left(\frac{\sqrt{3}}{2}\right)$.

Logo, para $\frac{y}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\frac{y}{2} = \text{arcsen}\left(\frac{\sqrt{3}}{2}\right) \iff \text{sen}\left(\frac{y}{2}\right) = \frac{\sqrt{3}}{2} \iff \frac{y}{2} = \frac{\pi}{3} \iff y = \frac{2\pi}{3}.$$

Exercício

Determine o domínio de $f(x) = \text{arcsen}(x-3)$.

Resolução:

$$
\begin{aligned}
D_f &= \{x \in \mathbb{R} : \ -1 \le x - 3 \le 1\} \\
&= \{x \in \mathbb{R} : \ 2 \le x \le 4\} \\
&= [2, 4] \, .
\end{aligned}
$$

Exercício

Mostre que $\cos(\text{arcsen}(x)) = \sqrt{1 - x^2}$, para $x \in (-1, +1)$.

Demonstração.

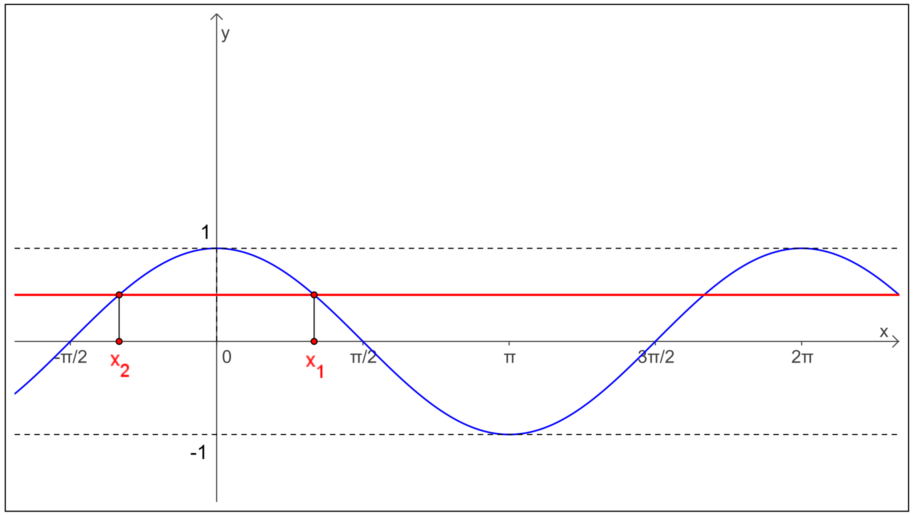
$$
\begin{aligned}
[\cos(\text{arcsen}(x))]^2 + [\text{sen}(\text{arcsen}(x))]^2 = 1 \quad &\Rightarrow \quad [\cos(\text{arcsen}(x))]^2 + x^2 = 1 \\
&\Rightarrow \quad [\cos(\text{arcsen}(x))]^2 = 1 - x^2 \\
&\Rightarrow \quad \sqrt{[\cos(\text{arcsen}(x))]^2} = \sqrt{1 - x^2} \\
&\Rightarrow \quad |\cos(\text{arcsen}(x))| = \sqrt{1 - x^2} \\
&\Rightarrow \quad \cos(\text{arcsen}(x)) = \sqrt{1 - x^2},
\end{aligned}
$$

pois se $x \in (-1, +1)$, então $\text{arcsen}(x) \in (-\pi/2, +\pi/2)$ e, assim, $\cos(\text{arcsen}(x)) > 0$. ■

A função arco cosseno

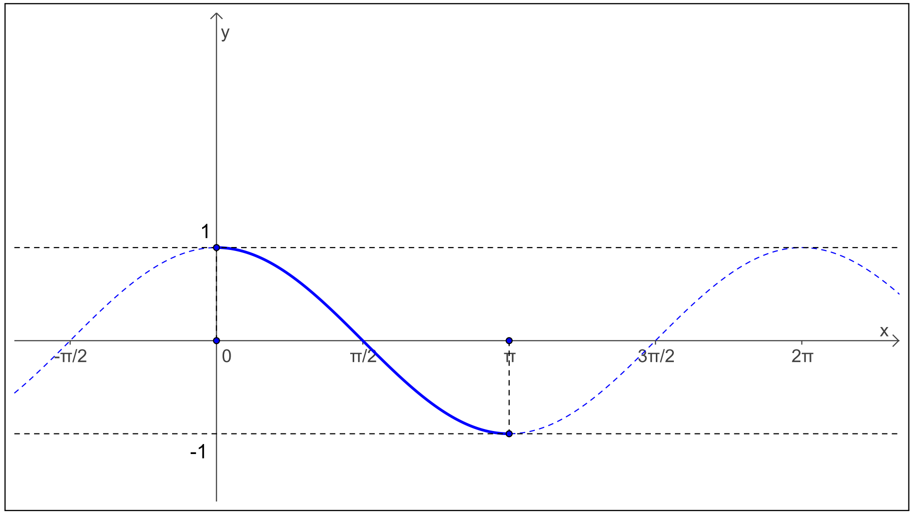
Função Cosseno

$$
\begin{array}{rccl}
f\colon & \mathbb{R} & \to & \mathbb{R} \\
& x & \mapsto & y = f(x) = \cos(x)
\end{array}
$$
não é inversível, pois não é injetiva.



Função Cosseno

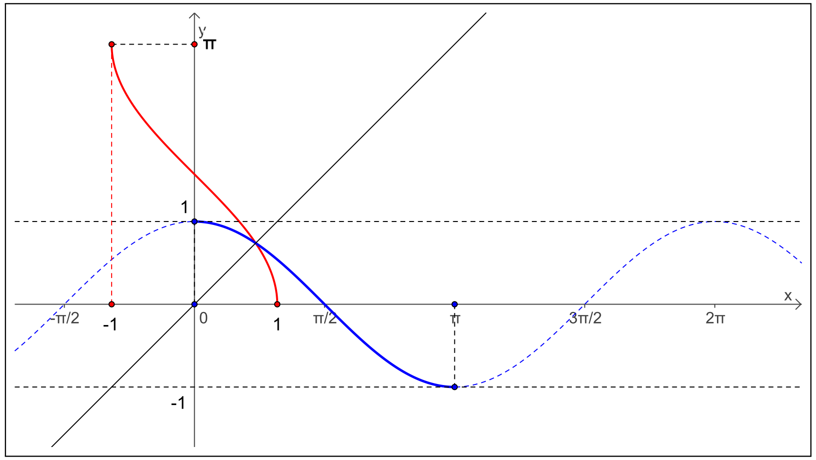
$$\begin{array}{rccc} f: & [0,\pi] & \to & [-1,+1] \\ & x & \mapsto & y = f(x) = \cos(x) \end{array}$$ é inversível, pois é bijetiva.



Função Arco Cosseno

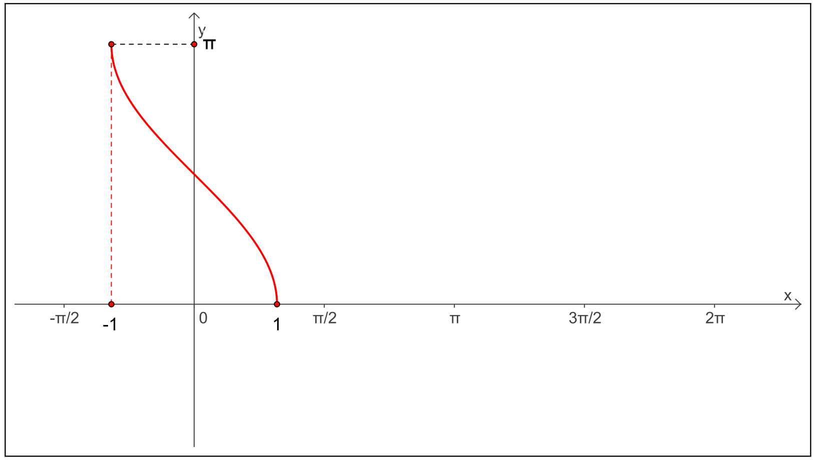
$$\begin{array}{rccc} f^{-1}: & [-1,+1] & \to & [0,\pi] \\ & x & \mapsto & y = f^{-1}(x) = \arccos(x) \end{array}$$

é a função arco cosseno, ou seja, a função inversa do cosseno.



Função arco cosseno: $\boxed{\begin{array}{rccc} f^{-1}: & [-1,+1] & \to & [0,\pi] \\ & x & \mapsto & y = f^{-1}(x) = \arccos(x) \end{array}}$

$\boxed{\text{Para } y \in [0,\pi], \quad y = \arccos(x), \text{ se e só se, } \cos(y) = x\,.}$



Exercício

a) Calcule $y = \arccos(\sqrt{3}/2)$. R.: $\pi/6$

b) Determine o domínio de $f(x) = \arccos(x^2 - 1)$. R.: $[-\sqrt{2}, \sqrt{2}]$

c) Determine o domínio de $f(x) = \operatorname{arcsen}(x - 3) + \arccos(x^2 - 10)$. R.: $[3, \sqrt{11}]$

Exercício

Mostre que $\operatorname{sen}(\arccos(x)) = \sqrt{1 - x^2}$, para $x \in (-1, +1)$.

Demonstração.

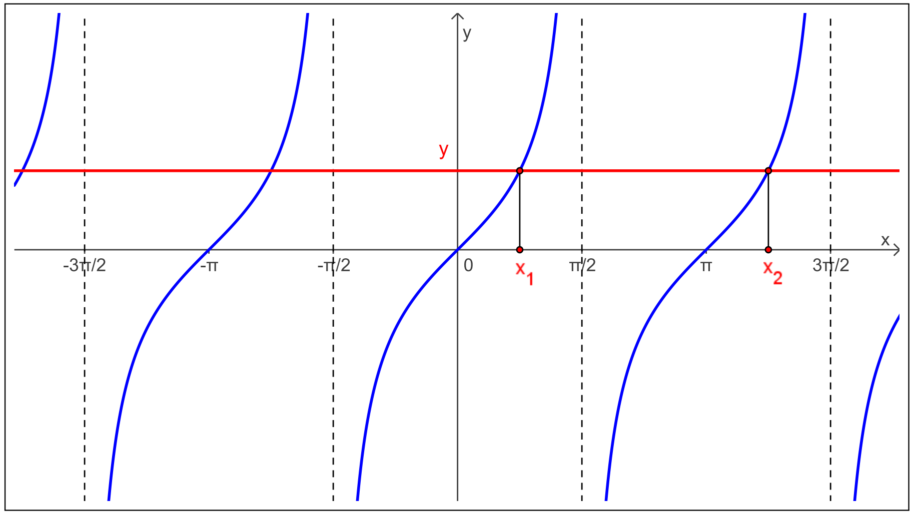
$$\begin{aligned}
[\cos(\arccos(x))]^2 + [\operatorname{sen}(\arccos(x))]^2 = 1 \quad &\Rightarrow \quad x^2 + [\operatorname{sen}(\arccos(x))]^2 = 1 \\
&\Rightarrow \quad [\operatorname{sen}(\arccos(x))]^2 = 1 - x^2 \\
&\Rightarrow \quad \sqrt{[\operatorname{sen}(\arccos(x))]^2} = \sqrt{1 - x^2} \\
&\Rightarrow \quad |\operatorname{sen}(\arccos(x))| = \sqrt{1 - x^2} \\
&\Rightarrow \quad \operatorname{sen}(\arccos(x)) = \sqrt{1 - x^2},
\end{aligned}$$

pois se $x \in (-1, +1)$, então $\arccos(x) \in (0, \pi)$ e, assim, $\operatorname{sen}(\operatorname{arcsen}(x)) > 0$. ■

A função arco tangente

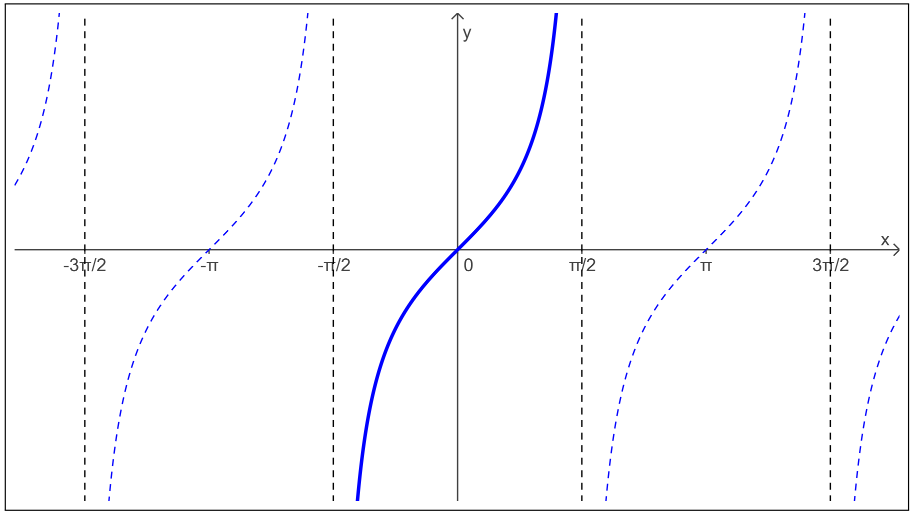
Função Tangente

$$\begin{array}{rccl} f\colon & \mathbb{R} - \{\pi/2 + k \cdot \pi \mid k \in \mathbb{Z}\} & \to & \mathbb{R} \\ & x & \mapsto & y = f(x) = \operatorname{tg}(x) \end{array}$$ não é inversível.



Função Tangente

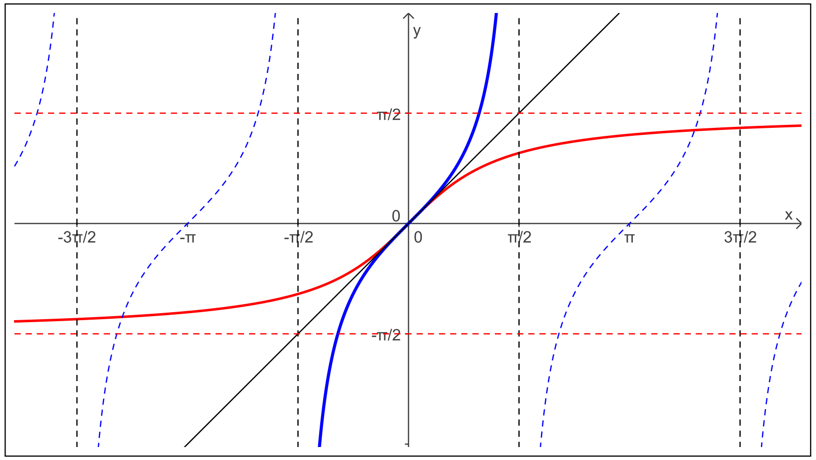
$$\begin{array}{rccl} f\colon & (-\pi/2, +\pi/2) & \to & \mathbb{R} \\ & x & \mapsto & y = f(x) = \operatorname{tg}(x) \end{array}$$ é inversível, pois é bijetiva.



Função Arco Tangente

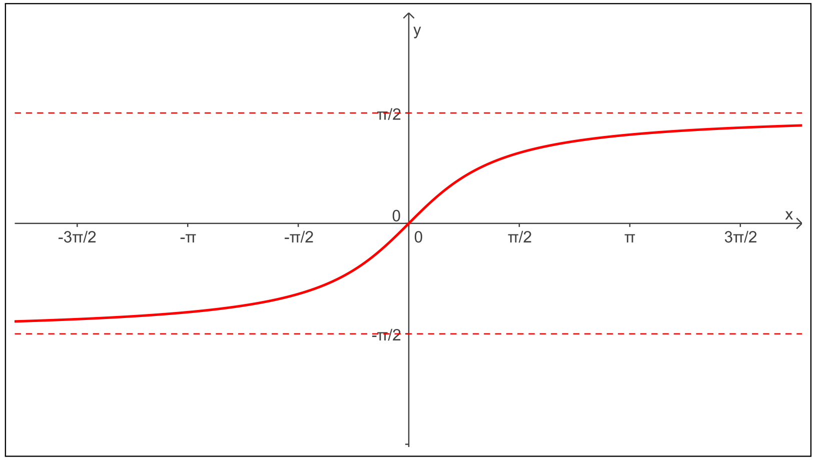
$$
\begin{array}{rccl}
f^{-1}: & \mathbb{R} & \to & (-\pi/2, +\pi/2) \\
 & x & \mapsto & y = f^{-1}(x) = \operatorname{arctg}(x)
\end{array}
$$

é a função arco tangente, ou seja, é a função inversa da tangente.



Função arco tangente: $\boxed{\begin{array}{rccl} f^{-1}: & \mathbb{R} & \to & (-\pi/2, +\pi/2) \\ & x & \mapsto & y = f^{-1}(x) = \operatorname{arctg}(x) \end{array}}$

$$\boxed{\text{Para } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y = \operatorname{arctg}(x), \text{ se e só se, } \operatorname{tg}(y) = x}\,.$$



Exercício

a) Calcule $\operatorname{arctg}(0)$. R.: 0

b) Calcule $\operatorname{sen}\left(\operatorname{arctg}\left(-\frac{\sqrt{3}}{3}\right)\right)$. R.: $-1/2$

c) Determine o domínio de $f(x) = \operatorname{arctg}\left(\frac{x^2-5}{x+3}\right)$. R.: $D = \mathbb{R} - \{-3\}$

Exercício

Mostre que $\sec^2(\operatorname{arctg}(x)) = 1 + x^2$, para $x \in \mathbb{R}$.

Demonstração.

$$[\cos(\operatorname{arctg}(x))]^2 + [\operatorname{sen}(\operatorname{arctg}(x))]^2 = 1$$
$$\Downarrow$$
$$\frac{[\cos(\operatorname{arctg}(x))]^2 + [\operatorname{sen}(\operatorname{arctg}(x))]^2}{\cos^2(\operatorname{arctg}(x))} = \frac{1}{\cos^2(\operatorname{arctg}(x))}$$
$$\Downarrow$$
$$1 + \operatorname{tg}^2(\operatorname{arctg}(x)) = \sec^2(\operatorname{arctg}(x))$$
$$\Downarrow$$
$$1 + x^2 = \sec^2(\operatorname{arctg}(x))$$
$$\Downarrow$$
$$\sec^2(\operatorname{arctg}(x)) = 1 + x^2.$$

■